TABLE II
$\|T_{\zeta v}\|_\infty$ AND $\|T_{u\omega}\|_2$ FOR THE MIXED $\mathcal{H}_2/\mathcal{H}_\infty$ AND REDUCED ORDER CONTROLLERS

	$\|T_{\zeta v}\|_\infty$	$\|T_{u\omega}\|_2$
$\mathcal{H}_2/\mathcal{H}_\infty$	1.292	22.6493
10 ord.	1.281	22.8842
3 ord.	1.292	24.8594

simpler SISO case, we anticipate that the results will extend naturally to multi-input multi-output problems.

Perhaps the most severe limitation of the proposed method is that it may result in very large order controllers (roughly $2N$), necessitating some type of model reduction. Note, however, that this disadvantage is shared by some widely used design methods, such as μ-synthesis or l_1 optimal control theory, that will also produce controllers with no guaranteed complexity bound. Application of some well-established methods in order reduction (noteworthy, weighted balanced truncation) usually succeed in producing controllers of manageable order. The example of Section IV suggests that substantial order reduction can be accomplished without performance degradation. Research is currently under way addressing this issue.

APPENDIX A
PROOF OF LEMMA 1

Proof of Lemma 1: From the maximum modulus theorem, it follows that a controller Q_i that is admissible for $\mathcal{H}_2/\mathcal{H}_{\infty,\,\delta_i}$ is also admissible for $\mathcal{H}_2/\mathcal{H}_{\infty,\,\delta_{i+1}}$. Thus, the sequence μ_i is nonincreasing, bounded below by the value of $\|T_{\zeta_2 w_2}\|_2$ obtained when using the optimal $\mathcal{H}_2$ controller. It follows then that it has a limit $\mu \geq \mu^o$. We will show next that $\mu = \mu^o$. Assume by contradiction that $\mu^o < \mu$ and select $\mu^o < \hat{\mu} < \mu$. Since $\inf_{Q \in \mathcal{RH}_\infty} \|R+Q\|_\infty < \gamma$, it follows that there exists $Q_1 \in \mathcal{RH}_\infty$ such that $\|R+Q_1\|_\infty < \gamma$. From the definition of μ^o it follows that, given $\eta > 0$, there exists $Q_o \in \mathcal{RH}_\infty$, $\|R+Q_o\|_\infty \leq \gamma$, such that $\|T_{\zeta_2 w_2}(Q_o)\|_2 \leq \mu^o + \eta$. Let $\hat{Q} \triangleq Q_o + \epsilon(Q_1 - Q_o)$. It follows that

$$\|T_{\zeta_2 w_2}(\hat{Q})\|_2 \leq \mu^o + \eta + \epsilon\|T_2(Q_1 - Q_o)\|_2$$
$$\|R + \hat{Q}\|_\infty \leq \epsilon\|R + Q_1\|_\infty + (1-\epsilon)\|R + Q_o\|_\infty < \gamma.$$

Since $\hat{Q} \in \mathcal{RH}_\infty$ it follows that there exists $\delta_1 < 1$ such that $T_1^\infty + T_2^\infty \hat{Q}$ is analytic in $|z| \geq \delta_1$. Since $\|T_1^\infty + T_2^\infty \hat{Q}\|_\infty < \gamma$, it follows from continuity that there exists $\delta_2 < 1$ such that $\|T_1^\infty + T_2^\infty \hat{Q}\|_{\infty,\,\delta_2} \leq \gamma$. Therefore, by taking ϵ and η small enough and $\delta \triangleq \max\{\delta_1, \delta_2\} < 1$ we have that $\|T_1^\infty + T_2^\infty \hat{Q}\|_{\infty,\,\delta} \leq \gamma$ and $\|T_{\zeta_2 w_2}(\hat{Q})\|_2 < \hat{\mu}$. Hence for $\delta_i \geq \delta$, $\mu_i < \hat{\mu}$. This contradicts the fact that the sequence μ_i is nonincreasing and that $\hat{\mu} < \mu = \lim_{\delta_i \to 1} \mu_i$. ◊

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$\mathcal{H}_\infty$ Control of Nonlinear Systems via Output Feedback: Controller Parameterization

Wei-Min Lu and John C. Doyle

***Abstract*— The standard state space solutions to the $\mathcal{H}_\infty$ control problem for linear time invariant systems are generalized to nonlinear time-invariant systems. A class of local nonlinear (output feedback) $\mathcal{H}_\infty$-controllers are parameterized as nonlinear fractional transformations on contractive, stable nonlinear parameters. As in the linear case, the $\mathcal{H}_\infty$ control problem is solved by its reduction to state feedback and output estimation problems, together with a separation argument. Sufficient conditions for $\mathcal{H}_\infty$-control problem to be locally solved are also derived with this machinery.**

I. INTRODUCTION

Linear $\mathcal{H}_\infty$ control theory has a simple state space characterization [3], which has clear connections with traditional methods in optimal control. These facts have stimulated several attempts to generalize the linear $\mathcal{H}_\infty$ results in state space to nonlinear systems [2], [13], [6],

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[1], [9], [5], [14], [8], [10]. We will use the accepted but unfortunate misnomer "nonlinear $\mathcal{H}_\infty$" to describe this research direction, which will be pursued further in this paper.

Our goal in this paper is to obtain a local $\mathcal{H}_\infty$ controller parameterization. Both plant and controllers are nonlinear time-invariant and realized as input-affine state-space equations. The $\mathcal{H}_\infty$-control problem is treated by the use of similar techniques in the linear case [3], and is solved by its reduction to state feedback and output estimation problems, together with a separation argument. Sufficient conditions for the output feedback $\mathcal{H}_\infty$-control problem to be locally solvable are also derived using this machinery. The solvability of the $\mathcal{H}_\infty$-control problem requires locally positive definite solutions to two Hamilton–Jacobi inequalities (HJI's) and these two solutions satisfy an additional condition. A class of local $\mathcal{H}_\infty$-controllers are parameterized as a nonlinear fractional transformation on locally contractive and stable nonlinear operators. In [9], a more comprehensive treatment for nonlinear $\mathcal{H}_\infty$-control problem is given.

The rest of this paper is organized as follows: in Section II, the $\mathcal{H}_\infty$-control problem is stated. In Section III, the simpler output estimation problem is considered. In Section II, the main results of this paper, local solutions to the output feedback $\mathcal{H}_\infty$-control problem, are given; the solvability of this problem requires the coupled positive definite solutions to two HI's, and a class of local $\mathcal{H}_\infty$-controllers are parameterized.

The following conventions are made in this paper. $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^+ := [0, \infty) \subset \mathbb{R}$. $\mathbb{R}^n$ is n-dimensional real Euclidean space; if $u \in \mathbb{R}^n$, then $\|u\|$ is Euclidean norm of u. $\mathbb{R}^{n\times m}$ is the set of real $n \times m$ matrices; if $A \in \mathbb{R}^{n\times m}$, then $A^T \in \mathbb{R}^{m\times n}$ is the transpose of A. A function is said to be of class $\mathbf{C}^k$ if it is continuously differentiable k times; so $\mathbf{C}^0$ stands for the class of continuous functions. $\|\cdot\|$ stands for the Euclidean norm. $\mathcal{B}_r := \{x \in \mathbb{R}^n \mid \|x\| < r, \text{ for some integer } n > 0\}$; we shall not specify the dimension of the environmental space, and always use the same r to denote its radius without confusion. $V: \mathbb{R}^n \to \mathbb{R}^+$ is locally positive-definite if there exists $r > 0$ such that for $x \in \mathcal{B}_r$, $V(x) = 0 \Rightarrow x = 0$; it is globally positive-definite if $V(x) = 0 \Rightarrow x = 0$, and $\lim_{x\to\infty} V(x) = \infty$. $\mathcal{L}_2[0, T]$, $\mathcal{L}_2[0, \infty)$ are two standard Lebesgue Spaces; $\mathcal{L}_2^e[0, \infty)$ is the extended space of $\mathcal{L}_2[0, \infty)$. $\Omega(G, K)$ represents fractional transformation of operator G on operator K; $\sum(M_1, M_2)$ stands for the Redheffer product of operators M_1 and M_2 (see [11], [9] for exact definitions).

II. Nonlinear $\mathcal{H}_\infty$-Control Problems

Consider the following input-affine nonlinear time-invariant (NLTI) system.

$$P: \begin{cases} \dot{x} = f(x) + g(x)w \\ z = h(x) + k(x)w \end{cases}$$

where $x \in \mathbb{R}^n$ is state vector and $w \in \mathbb{R}^p$ and $z \in \mathbb{R}^q$ are input and output vectors, respectively. We will assume f, g, h, $k \in \mathbf{C}^0$, and $f(0) = 0$, $h(0) = 0$. Therefore, $0 \in \mathbb{R}^n$ is the equilibrium of the system with $w = 0$. The state transition function $\phi: \mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{L}_2[0, \infty] \to \mathbb{R}^n$ is so defined that $x = \phi(T, x_0, w^*)$ means that system P evolves from initial state x_0 to state x in time T under the control action w^*.

Definition 2.1:

i) System P (or $[f(x), g(x)]$) is reachable from zero if for all $x \in \mathbb{R}^n$, there exist $T \in R^+$ and $w^*(t) \in \mathcal{L}_2[0, T]$ such that $x = \phi(T, 0, w^*)$;

ii) System P (or $[h(x), f(x)]$) is (zero-state) detectable if for all $x \in \mathbb{R}^n$, $h(\phi(t, x, 0)) = 0 \Rightarrow \phi(t, x, 0) \to 0$ as $t \to \infty$.

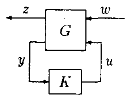

Fig. 1.

Definition 2.2: System P is said to have $\mathcal{L}_2$-gain less than or equal to γ for some $\gamma > 0$ if

$$\int_0^T \|z(t)\|^2 dt \le \gamma^2 \int_0^T \|w(t)\|^2 dt$$

for all $T > 0$ and $w(t) \in \mathcal{L}_2[0, T]$, and $z(t) = h(\phi(t, 0, w(t)) + k(\phi(t, 0, w(t))w(t)$.

The following result characterizes the $\mathcal{L}_2$-gain for a class of nonlinear systems [13], [9].

Proposition 2.1: Consider system P with $R(x) := I - k^T(x)k(x) > 0$ for all $x \in \mathbb{R}^n$, if there is a $\mathbf{C}^1$ function $V: \mathbb{R}^n \to \mathbb{R}^+$ with $V(0) = 0$ such that

$$\begin{aligned} \mathcal{H}(V, x) := {} & \frac{\partial V}{\partial x}(x)(f(x) - g(x)R^{-1}(x)k^T(x)h(x)) \\ & + \frac{1}{4}\frac{\partial V}{\partial x}(x)g(x)R^{-1}(x)g^T(x)\frac{\partial V^T}{\partial x}(x) \\ & + h^T(x)(I - k(x)k^T(x))^{-1}h(x) \le 0. \qquad (1) \end{aligned}$$

then $\dot{V}(x) \le \|w\|^2 - \|z\|^2$; moreover, P has $\mathcal{L}_2$ – gain ≤ 1.

In the following, we denote $\mathcal{FG}$ as the class of all input-affine NLTI systems which are asymptotically stable with zero input and related HJI $\mathcal{H}(V, x) \le 0$ has a positive definite solution.

Next, we state the $\mathcal{H}_\infty$-control problem. The feedback configuration for the $\mathcal{H}_\infty$-control synthesis problem is depicted in Fig. 1, where G is a nonlinear plant with two sets of inputs: the exogenous disturbance input w and the control input u, and two sets of outputs: the measured output y and the regulated output z. K is the controller to be designed. It is required that the closed-loop system, which is the fractional transformation of G on K and denoted as nonlinear operator $\Omega(G, K)$, be well posed. Both G and K are nonlinear time-invariant and can be realized as control-affine state-space equations. That is

$$G: \begin{cases} \dot{x} = f(x) + g_1(x)w + g_2(x)u \\ z = h_1(x) + k_{11}(x)w + k_{12}(x)u \\ y = h_2(x) + k_{21}(x)w + k_{22}(x)u \end{cases}$$

where f, g_i, h_i, $k_{ij} \in \mathbf{C}^2$ and $f(0) = 0$, $h_1(0) = 0$, $h_2(0) = 0$; x, w, u, z, and y are assumed to have dimensions n, p_1, p_2, q_1, and q_2, respectively.

$$K: \begin{cases} \dot{\hat{x}} = a(\hat{x}) + b(\hat{x})y \\ u = c(\hat{x}) + d(\hat{x})y \end{cases}$$

with a, b, c, $d \in \mathbf{C}^2$ and $a(0) = 0$, $c(0) = 0$.

The initial states for both plant and controller are $x(0) = 0$ and $\hat{x}(0) = 0$. We shall consider the following output feedback (OF) $\mathcal{H}_\infty$-control problem.

$\mathcal{H}_\infty$-Control Problem: Find an output feedback controller K (or a class controllers) if any, such that the closed-loop system $\Omega(G, K)$ is asymptotically stable with $w = 0$ and has $\mathcal{L}_2$ – gain ≤ 1, i.e.,

$$\int_0^T (\|w(t)\|^2 - \|z(t)\|^2) dt \ge 0$$

for all $T \in \mathbb{R}^+$.

The following assumptions on system structure are made:

A1) $k_{11}(x) = 0$, $k_{22}(x) = 0$,

A2) $k_{12}^T(x)[h_1(x)\ k_{12}(x)] = [0\ I]$, A3) $\begin{bmatrix} g_1(x) \\ h_{21}(x) \end{bmatrix} k_{21}^T(x) = \begin{bmatrix} 0 \\ I \end{bmatrix}.$

It is noted that, when the $\mathcal{H}_\infty$-control problem is considered, many nonlinear systems can be transformed into systems with the above structural constraints as in the linear case [12], [3].

III. $\mathcal{H}_\infty$-Control Synthesis: Output Estimation

The $\mathcal{H}_\infty$-control problem is solved by its reduction into a simpler problem, which is called output estimation (OE) problem. The system in this case has the following structure

$$G_{\rm OE}: \begin{cases} \dot{x} = f(x) + g_1(x)w + g_0(x)u \\ z = h_1(x) + u \\ y = h_2(x) + k_{21}(x)w. \end{cases}$$

The assumption for this structure is

A3) $\begin{bmatrix} g_1(x) \\ k_{21}(x) \end{bmatrix} k_{21}^T(x) = \begin{bmatrix} 0 \\ I \end{bmatrix}.$

The OE $\mathcal{H}_\infty$-controllers are constructed by using the similar idea to the linear case in [3]; the detailed construction and the consideration of stability issue are given in [9].

Theorem 3.1: Consider $G_{\rm OE}$; suppose there exists a $\mathbf{C}^3$ positive definite solution $U(x)$ to HJI

$$\begin{aligned} \mathcal{H}_{FC}(U, x) := & \frac{\partial U}{\partial x}(x) f(x) \\ & + \frac{1}{4}\frac{\partial U}{\partial x}(x) g_1(x) g_1^T(x) \frac{\partial U^T}{\partial x}(x) \\ & + h_1^T(x) h_1(x) - h_2^T(x) h_2(x) \leq 0 \end{aligned}$$

with $U(0) = 0$, and $U(x)$ makes the Hessian matrix of $\mathcal{H}_{\rm FC}(U, x)$ with respect to $x \in \mathbf{R}^n$ be negative definite at zero. If a $\mathbf{C}^2$ matrix-valued function $L_0(x)$ satisfies

$$\frac{\partial U}{\partial x}(x) L_0(x) = -2h_2^T(x)$$

then there is a controller which makes the closed-loop system have $\mathcal{L}_2$–gain ≤ 1; such a controller is given by

$$K_{\rm OE}: \begin{cases} \dot{\tilde{x}} = f(\tilde{x}) - g_0(\tilde{x})h_1(\tilde{x}) + L_0(\tilde{x})h_2(\tilde{x}) - L_0(\tilde{x})y \\ u = -h_1(\tilde{x}) \end{cases}.$$

The following two lemmas are required in the proof; their proofs are given in [9]

Lemma 3.2: Suppose $U(x)$ and $L_0(x)$ are given in Theorem 3.1, define

$$\begin{aligned} \mathcal{H}_{\rm OI}(U, L_0, x) := & \frac{\partial U}{\partial x}(x)(f(x) + L_0(x)h_2(x)) \\ & + \frac{1}{4}\frac{\partial U}{\partial x}(x)(g_1(x) + L_0(x)k_{21}(x))(g_1(x) \\ & + L_0(x)k_{21}2(x))^T \frac{\partial U^T}{\partial x}(x) + h_1^T(x)h_1(x) \end{aligned}$$

then $\mathcal{H}_{\rm OI}(U, L_0, x) = \mathcal{H}_{\rm FC}(U, x)$.

Lemma 3.3: Suppose that $U(x)$ and $L_0(x)$ are defined as in Theorem 3.1. Let x, $\tilde{x}$ be states of systems $G_{\rm OE}$ and $K_{\rm OE}$, $e = \tilde{x} - x$. Define

$$\begin{aligned} \mathcal{H}_e(e, \tilde{x}) := & \frac{\partial U}{\partial e}(e)(f(\tilde{x}) - f(x) + L_0(\tilde{x})(h_2(\tilde{x}) - h_2(x))) \\ & + \frac{1}{4}\frac{\partial U}{\partial e}(e)(g_1(x) + L_0(\tilde{x})k_{12}(x))(g_1(x) \\ & + L_0(\tilde{x})k_{12}(x))^T \frac{\partial U^T}{\partial e}(e) \\ & + (h_1^T(\tilde{x}) - h_1^T(x))(h_1(\tilde{x}) - h_1(x)) \\ & - \frac{\partial U}{\partial e}(e)(g_0(x) - g_0(\tilde{x}))h_1(\tilde{x}). \end{aligned}$$

Then for all $(x, \tilde{x}) \in \beta_r$ with some $r > 0$, $\mathcal{H}_e(e, \tilde{x}) \leq 0$.

Note that if $\mathcal{H}_e(e, \tilde{x}) \leq 0$, then

$$\begin{aligned} & \frac{\partial U}{\partial e}(e)((f(\tilde{x}) - f(x) + L_0(\tilde{x})(h_2(\tilde{x}) - h_2(x)) \\ & - (L_0(\tilde{x})k_{21}(x) + g_1(x))w) \\ & \leq \frac{\partial U}{\partial e}(e)(g_0(\tilde{x}) - g_0(x))h_1(\tilde{x}) \\ & + \|w\|^2 - \|h_1(x) - h_1(\tilde{x})\|^2. \end{aligned}$$

Now, we prove Theorem 3.1 based on the above observation.

Proof: Consider $\Omega(G_{\rm OE}, K_{\rm FC})$ which has following realization

$$\begin{cases} \dot{x} = f(x) - g_0(x)h_1(\tilde{x}) + g_1(x)w \\ \dot{\tilde{x}} = f(\tilde{x}) - g_0(\tilde{x})h_1(\tilde{x}) + L_0(\tilde{x})(h_1(\tilde{x}) - h_2(x) - k_{21}(x)w) \\ z = h_1(x) - h_1(\tilde{x}). \end{cases}$$

Let $e = \tilde{x} - x$, for $(x, \tilde{x}) \in \beta_r$.

$$\begin{aligned} \dot{U}(e) = & \frac{\partial U}{\partial e}(e)((f(\tilde{x}) - f(x) + L_0(\tilde{x})(h_2(\tilde{x}) - h_2(x)) \\ & - (L_0(\tilde{x})k_{21}(x) + g_1(x)w) - \frac{\partial U}{\partial e}(e)(g_0(\tilde{x}) \\ & - g_0(x))h_1(\tilde{x}) \\ \leq & -\|z\|^2 + \|w\|^2. \end{aligned}$$

the last inequality follows from the preceding lemma. Thus

$$\int_0^T (\|w\|^2 - \|z\|^2)dt \geq U(e(T)) - U(0) = U(e(T)) \geq 0.$$

for all $T \geq 0$, which implies $\mathcal{L}_2$–gain ≤ 1. □

Theorem 3.4: Under the assumption of Theorem 3.1, if in addition, $L_1(x)$ is such that $\partial U/\partial x(x) L_1(x) = -2h_1^T(x)$, then the controller $u = \Omega(M_{\rm OE}, Q)y$ with $M_{\rm OE}$ given by

$$\begin{cases} \dot{\tilde{x}} = f(\tilde{x}) - g_0(\tilde{x})h_1(\tilde{x}) + L_0(\tilde{x})h_2(\tilde{x} - y) + (g_2(\tilde{x}) + L_1(\tilde{x}))u_0 \\ u = -h_1(\tilde{x}) + u_0 \\ y_0 = h_2(\tilde{x}) - y \end{cases}$$

for all $Q \in \mathcal{FG}$ also makes the closed-loop system (locally) has $\mathcal{L}_2$ – gain ≤ 1.

Proof: Consider $\Omega(G_{\rm OE}, \Omega(M_{\rm OE}, Q))$ for $Q \in \mathcal{FG}$ which has following realization.

$$Q: \begin{cases} \dot{\xi} = a(\xi) + b(\xi)y_0 \\ u_0 = c(\xi) \end{cases}.$$

Let U_Q be a solution to the HJI with respect to Q with state ξ, then $\dot{U}_Q(\xi) \leq \|y_0\|^2 - \|u_0\|^2$. The similar argument shows that there exists $r > 0$, for $(x, \hat{x}, \xi) \in \mathcal{B}_r$

$$\dot{U}(e) \leq \|w\|^2 - \|z\|^2 - \|y_0\|^2 + \|u_0\|^2.$$

Thus

$$\dot{U}(e) + \dot{U}_Q(\xi) \leq -\|z\|^2 + \|w\|^2 \leq -\|z\|^2 + \|w\|^2.$$

Therefore

$$\int_0^T (\|z\|^2 - \|w\|^2)dt \leq U(0) - U(e(T)) = -U(e(T)) \leq 0$$

for all $T \in \mathbf{R}^+$, which implies the $\mathcal{L}_2$ – gain ≤ 1. □

IV. $\mathcal{H}_\infty$-Control Problems: Main Results

We now consider the general $\mathcal{H}_\infty$ control problem. The nonlinear time-invariant plant is realized as control-affine state-space equation

$$G: \begin{cases} \dot{x} = f(x) + g_1(x)w + g_2(x)u \\ z = h_1(x) + k_{12}(x)u \\ y = h_2(x) + k_{21}(x)w \end{cases}$$

where $f(0) = 0$, $h_1(0) = 0$, $h_2(0) = 0$; x, w, u, z, and y are assumed to have dimensions n, p_1, p_2, q_1, and q_2, respectively.

The following assumptions are made

A2) $k_{12}^T(x)[h_1(x) \quad k_{12}(x)] = [0 \quad I]$,

A3) $\begin{bmatrix} g_1(x) \\ k_{21}(x) \end{bmatrix} k_{21}^T(x) = \begin{bmatrix} 0 \\ I \end{bmatrix}$,

A4) $[h_1(x), \ f(x)]$ is zero-state detectable.

The following two quantities are defined

$$\mathcal{H}_{\rm FI}(V, x) := \frac{\partial V}{\partial x}(x)f(x) + h_1^T(x)h_1(x) + \frac{1}{4}\frac{\partial V}{\partial x}(x)(g_1(x)g_1^T(x) - g_2(x)g_2^T(x))\frac{\partial V^T}{\partial x}(x) \tag{2}$$

$$\mathcal{H}_{\rm FC}(U, x) := \frac{\partial U}{\partial x}(x)f(x) + \frac{1}{4}\frac{\partial U}{\partial x}(x)g_1(x)g_1^T(x)\frac{\partial U^T}{\partial x}(x) + h_1^T(x)h_1(x) - h_2^T(x)h_2(x). \tag{3}$$

The main idea of construction is to convert the general problem OF into the simpler problems which have been solved.

Let $V(x) \geq 0$ be the solution of $\mathcal{H}_{\rm FI}(V, x) \leq 0$. Define

$$F_0(x) := -\frac{1}{2}g_2^T(x)\frac{\partial V^T}{\partial x}(x), \quad F_1(x) := \frac{1}{2}g_1^T(x)\frac{\partial V^T}{\partial x}(x).$$

Let $r := w - F_1(x)$ and $v := u - F_0(x)$. After the change of variables, the original plant G becomes G_a as follows

$$G_a: \begin{cases} \dot{x} = f_a(x) + g_1(x)r + g_2(x)u \\ v = h_a(x) + u \\ y = h_2(x) + k_{21}(x)r \end{cases}$$

where $f_a(x) := f(x) + g_1(x)F_1(x)$, $h_a(x) := F_0(x)$.

Lemma 4.1: Consider systems G and G_a. If the controller K makes $\Omega(G_a, K)$ have $\mathcal{L}_2$–gain ≤ 1, it also results in $\Omega(G, K)$ having $\mathcal{L}_2$–gain ≤ 1.

Proof: Note that $z = \Omega(G, K)w$ and $r = \Omega(G_a, K)v$.

Since $V(x) \geq 0$ satisfies $\mathcal{H}_{\rm FI}(V, x) \leq 0$, let $\Psi(x) \geq 0$ be such that $\mathcal{H}_{\rm FI}(V, x) + \psi(x) = 0$, then

$$\begin{aligned} \dot{V}(x) &= \frac{\partial V}{\partial x}(x)(f(x) + g_1(x)w + g_2(x)u) \\ &= -\|z\|^2 + \|w\|^2 - \left\| w - \frac{1}{2}g_1^T(x)\frac{\partial V^T}{\partial x}(x)\right\|^2 \\ &\quad + \left\| u + \frac{1}{2}g_2^T(x)\frac{\partial V^T}{\partial x}(x)\right\|^2 - \psi(x) \\ &\leq -\|z\|^2 + \|w\|^2 - \|v\|^2 + \|r\|^2. \end{aligned}$$

So for all $T \geq 0$

$$\int_0^T (\|w\|^2 + \|z\|^2)dt \geq \int_0^T (\|v\|^2 - \|r\|^2)dt + V(x(T)) \geq \int_0^T (\|v\|^2 - \|r\|^2)dt.$$

$$\int_0^T (\|v\|^2 - \|r\|^2)dt \geq 0 \Rightarrow \int_0^T (\|w\|^2 - \|z\|^2)dt \geq 0.$$

□

Note that system G_a is of OE structure and satisfies the structure assumption A3). Define

$$\mathcal{H}_a(W, x) := \frac{\partial W}{\partial x}(x)f_a(x) + \frac{1}{4}\frac{\partial W}{\partial x}(x)g_1(x)g_1^T(x)\frac{\partial W^T}{\partial x}(x) + h_a^T(x)h_a(x) - h_2^T(x)h_2(x).$$

Take $W(x) = U(x) - V(x)$ with $W(0) = U(0) - V(0)$ where $V(x) \geq 0$ is given just now. Note that

$$\mathcal{H}_a(W, x) = \mathcal{H}_{\rm FC}(U, x) - \mathcal{H}_{\rm FI}(V, x) = \mathcal{H}_{\rm FC}(U, x) + \psi(x)$$

where $\psi(x) \geq 0$ is such that $\mathcal{H}_{\rm FI}(V, x) + \psi(x) = 0$. Thus, $\mathcal{H}_a(W, x) \leq 0$ if and only if $\mathcal{H}_{\rm FC}(U, x) + \psi(x) \leq 0$. Assume $U(x)$ is such that $\mathcal{H}_{\rm FC}(U, x) + \psi(x) \leq 0$ has a negative definite Hessian matrix at $x = 0$, then $\mathcal{H}_a(W, x)$ also has negative definite Hessian matrix at $x = 0$. Suppose $L_0(x)$ is such that $\partial W/\partial x(x)L_0(x) = -2h_2^T(x)$. The controller K for the new OE structure given by Theorem 3.1

$$K: \begin{cases} \dot{\hat{x}} = f_a(\hat{x}) + g_2(\hat{x})h_a(\hat{x}) + L_0(\hat{x})h_2(\hat{x}) - L(\hat{x})y \\ u = h_a(\hat{x}) \end{cases}$$

is such that system $\Omega(G_a, K)$ locally has $\mathcal{L}_2$–gain ≤ 1.

By Lemma 4.1, $\Omega(G, K)$ has $\mathcal{L}_2$–gain ≤ 1. Next, we examine the stability of the closed-loop system $\Omega(G, K)$ which has the following realization

$$\begin{cases} \dot{x} = f(x) + g_2(x)F_0(\hat{x}) + g_1(x)w \\ \dot{\hat{x}} = f_K(\hat{x}) + L_0(\hat{x})(h_2(\hat{x}) - h_2(x)) + L_0(\hat{x})k_{21}(x)w \\ z = h_1(x) + k_{12}(x)F_0(\hat{x}) \end{cases}$$

where

$$f_K(\hat{x}) := f(\hat{x}) + g_1(\hat{x})F_1(\hat{x}) + g_2(\hat{x})F_0(\hat{x}). \tag{4}$$

Take $e = \hat{x} - x$. Note that $\mathcal{H}_a(W, \cdot)$ has negative definite Hessian matrix as does $\mathcal{H}_{\rm FC}(U, \cdot)$. Moreover, there is some locally positive definite $\pi: \mathbb{R}^n \to \mathbb{R}^+$, such that if $(x, \hat{x}) \in \mathrm{B}_s$ for some $s \geq 0$, then $\mathcal{H}_a(W, e) + \pi(e) \leq 0$; moreover

$$\dot{W}(e) \leq \|r\|^2 - \|v\|^2 - \pi(e).$$

Let $L_{\rm OF}(x, e) = V(x) + W(e)$ with $e = \hat{x} - x$. By assumption $V(x)$ and $W(e)$ are positive definite, so is $L_{\rm OF}(x, e)$, and it can be used as a Lyapunov function. Take $w = 0$

$$\dot{V}(x) \leq -\|z\|^2 + \|v\|^2 - \|r\|^2$$

$$\dot{L}_{\rm OF}(x, e) = \dot{V}(x) + \dot{W}(e) \leq -\|z\|^2 - \pi(e) \leq 0.$$

Then $\dot{L}_{\mathrm{OF}}(x,\ e) = 0 \Rightarrow z = 0$ and $\pi(e) = 0 \Rightarrow x = 0$ [assumption A4)] and $e = 0$. Thence, $L_{\mathrm{OF}}(x,\ e)$ is locally negative definite, the closed-loop system is thus locally asymptotically stable.

Therefore, we have the following result about output $\mathcal{H}_\infty$-control problem, an equivalent version of which was obtained by Isidori earlier in [5].

Theorem 4.2: Consider G, if there is some $\psi(x) \geq 0$ with $\psi(0) = 0$ such that

i) There exists a positive definite $V(x)$ which solves the Hamilton–Jacobi equation $\mathcal{H}_{\mathrm{FI}}(V,\ x) + \psi(x) = 0$ with $V(0) = 0$.

ii) there exists a positive definite $U(x)$ which satisfies the HJI: $\mathcal{H}_{\mathrm{FC}}(U,\ x) + \psi(x) \leq 0$ with $U(0) = 0$. And $\mathcal{H}_{\mathrm{FC}}(U,\ x) + \psi(x)$ has nonsingular Hessian matrix at zero.

iii) $U(x) - V(x) \geq 0$ is positive definite. And

$$\left(\frac{\partial U}{\partial x}(x) - \frac{\partial V}{\partial x}(x)\right)L_0(x) = -2h_2^T(x)$$

has a solution $L_0(x)$. Then the $\mathcal{H}_\infty$-control problem is (locally) solvable, and such a controller is given by

$$K: \begin{cases} \dot{\hat{x}} = f_K(\hat{x}) + L_0(\hat{x})h_2(\hat{x}) - L_0(\hat{x})y \\ u = F_0(\hat{x}). \end{cases}$$

Note that $\mathcal{H}_\infty$-controllers have separation structures. The separation principle for the $\mathcal{H}_\infty$-performance in nonlinear systems was confirmed by Ball–Helton–Walker [1] (see also [5]). The following result gives a $\mathcal{H}_\infty$-controller parameterization.

Theorem 4.3: Consider a system G satisfying the condition in Theorem 5.1. If in addition $L_1(x)$ satisfies

$$\left(\frac{\partial U}{\partial x}(x) - \frac{\partial V}{\partial x}(x)\right)L_1(x) = -2h_1^T(x)$$

then the controller $u = \Omega(M,\ Q)y$ with M given by

$$\begin{cases} \dot{\tilde{x}} = f_K(\tilde{x}) - L_0(\tilde{x}) \quad y + (g_2(\tilde{x}) + L_1(\tilde{x}))\,u_0 \\ u = F_0(\tilde{x}) + u_0 \\ \quad y_0 = h_2(\tilde{x}) - y \end{cases}$$

for all $Q \in \mathcal{FG}$ also (locally) solves OF $\mathcal{H}_\infty$-control problem.

Proof: By Lemma 4.1 and Theorem 3.4, it follows that the closed-loop system $\Omega(G,\ K)$ with $K = \Omega(M,\ Q)$ has $\mathcal{L}_2$-gain ≤ 1. Now it is sufficient to consider the stability issue. Suppose Q has the following realization

$$Q: \begin{cases} \dot{\xi} = a(\xi) + b(\xi)y_0 \\ u_0 = c(\xi) \end{cases}$$

and $U_Q(\xi)$ is such that $\dot{U}_Q(\xi) \leq \|u_0\|^2 - \|y_0\|^2$. Take $w = 0$, the closed-loop system has following hierarchical structure

$$\begin{cases} \dot{x} = f(x) + g_2(x)(F_0(\tilde{x}) + c(\xi)) \\ \dot{\xi} = a(\xi) + b(\xi)(h_2(\tilde{x}) - h_2(x)) \\ \dot{\tilde{x}} = f_K(\tilde{x}) + L_0(\tilde{x})(h_2(\tilde{x}) - h_2(x)) + (g_2(\tilde{x}) + L_1(\tilde{\xi}))c(\xi). \end{cases}$$

Let $V,\ W: \mathbf{R}^n \rightarrow \mathbf{R}^+$ be positive definite and defined as in the preceding discussion. Denote $e = \tilde{x} - x$. Similar arguments to Theorems 3.4 and 4.2 show that

$$\dot{W}(e) \leq \|r\|^2 - \|v\|^2 - \|y_0\|^2 + \|u_0\| - \pi(e)$$

for some positive definite $\pi: \mathbf{R}^n \rightarrow \mathbf{R}^+$. Define $L_{\mathrm{OF}}(x,\ e,\ \xi) := V(x) + W(e) + U_Q(\xi)$ as the Lyapunov function of the closed-loop system, then $\dot{L}_{\mathrm{OF}}(x,\ e,\ \xi) \leq -\|z\|^2 - \pi(e)$. Now $\dot{L}_{\mathrm{OF}}(e,\ \xi) = 0 \Rightarrow \pi(e) = 0$ and $\|z\| = 0$, so $e = 0$ and $z = 0$; the latter implies $x(t) \rightarrow 0$ as $t \rightarrow \infty$ by A4); on the other hand, if $e = 0$, $x = 0$, then $\dot{\xi} = a(\xi)$, which is asymptotically stable and $\xi(t) \rightarrow 0$ as $t \rightarrow \infty$. The interconnected system is locally asymptotically stable by LaSelle's theorem and Vidyasagar's theorem [15]. □

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