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Computation of μ with Real and Complex Uncertainties

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Abstract

The robustness analysis of system performance is one of the key issues in control theory, and one approach is to reduce this problem to that of computing the structured singular value, μ. When real parametric uncertainty is included, then μ must be computed with respect to a block structure containing both real and complex uncertainties, and this is the situation considered here. It is shown that μ is equivalent to a real eigenvalue maximization problem, and a power algorithm is developed to solve this problem. The algorithm has the property that μ is (almost) always an equilibrium point of the algorithm, and that whenever the algorithm converges a lower bound for μ results. Numerical experience with the algorithm is very encouraging.

1 Introduction

The structured singular value, μ, was introduced in [1] as a tool for robustness analysis. Its usefulness lies in the fact that many robustness problems can be re-cast as one of computing μ with respect to some block structure. In general, it is difficult to compute μ exactly, so computation has focused on upper and lower bounds. Such bounds are useful in their own right however, since an upper bound gives a (possibly conservative) limit on the size of allowable perturbations, and a lower bound yields a "problem perturbation". Important issues then become the efficient computation of the bounds, the degree to which they approximate μ, and techniques for refining the bounds for a better approximation.

For the purely complex case (i.e. the block structure contains only complex uncertainties) a tractable upper bound was suggested in [1] involving a singular value minimization. Computation schemes for lower bounds have been developed involving a smooth optimization problem, due to Fan and Tits [2], and a power algorithm, due to Packard et. al. [3]. Whilst the purely complex case is by no means completely solved, these methods are now routinely applied to large engineering problems.

The mixed case (i.e. the block structure contains both real and complex uncertainties) however is a fundamentally more difficult problem, and is much less well understood. An upper bound was recently presented by Fan et. al. [4] which involves minimizing the eigenvalues of a Hermitian matrix. This paper addresses the problem of computing a lower bound for μ in the mixed case. It is shown that μ can be obtained as the result of a (non-convex) real eigenvalue maximization, and this problem can be tackled by means of a power algorithm. The power algorithm is an extension to the mixed case of Packard's algorithm [3], which in turn is an extension of standard power iterations for eigenvalues and singular values. In general, there is no guarantee that the global maximum has been found and hence a lower bound for μ results.

2 Notation and Preliminaries

The notation used here is fairly standard and is essentially taken from [4]. For any square complex matrix M we denote the complex conjugate transpose by M^*. The largest singular value and the structured singular value are denoted by $\overline{\sigma}(M)$ and $\mu_{\mathcal{K}}(M)$ respectively. The spectral radius is denoted $\rho(M)$ and $\rho_R(M) = max\{|\lambda| : \lambda \textit{ is a real eigenvalue of } M\}$, with $\rho_R(M) = 0$ if M has no real eigenvalues. For any complex vector x, then x^* denotes the complex conjugate transpose, $|x|$ the Euclidean norm, and $\|x\|_\infty$ the infinity norm.

The definition of μ is dependent upon the underlying block structure of the uncertainties, which is defined as follows. Given a matrix $M \in \mathbf{C}^{n\times n}$ and three non-negative integers m_r, m_c, and m_C with $m := m_r + m_c + m_C \leq n$, the block structure $\mathcal{K}(m_r, m_c, m_C)$ is an m-tuple of positive integers

$$\mathcal{K} = (k_1, \ldots, k_{m_r}, k_{m_r+1}, \ldots, k_{m_r+m_c}, k_{m_r+m_c+1}, \ldots, k_m) \tag{1}$$

where we require $\sum_{i=1}^{m} k_i = n$ in order that the dimensions are compatible. This now determines the set of allowable perturbations, namely define

$$\begin{aligned} X_{\mathcal{K}} = \{\Delta = block\ diag(&\delta_1^r I_{k_1}, \ldots, \delta_{m_r}^r I_{k_{m_r}}, \delta_1^c I_{k_{m_r+1}}, \ldots, \\ &\delta_{m_c}^c I_{k_{m_r+m_c}}, \Delta_1^C, \ldots, \Delta_{m_C}^C) : \\ &\delta_i^r \in \mathbf{R}, \delta_i^c \in \mathbf{C}, \Delta_i^C \in \mathbf{C}^{k_{m_r+m_c+i}\times k_{m_r+m_c+i}}\} \end{aligned} \tag{2}$$

Note that $X_{\mathcal{K}} \in \mathbf{C}^{n\times n}$ and that this block structure is sufficiently general to allow for repeated real scalars, repeated complex scalars, and full complex blocks. The purely complex case corresponds to $m_r = 0$.

Definition 1 ([1]) *The structured singular value, $\mu_{\mathcal{K}}(M)$, of a matrix $M \in \mathbf{C}^{n\times n}$ with respect to a block structure $\mathcal{K}(m_r, m_c, m_C)$ is defined as*

$$\mu_{\mathcal{K}}(M) = \left(\min_{\Delta\in X_{\mathcal{K}}} \{\overline{\sigma}(\Delta) : det(I - \Delta M) = 0\}\right)^{-1} \tag{3}$$

with $\mu_{\mathcal{K}}(M) = 0$ if no $\Delta \in X_{\mathcal{K}}$ solves $det(I - \Delta M) = 0$.

Whilst it is not at all obvious how to compute μ from the definition (3) it is easy to obtain the crude bounds

$$\rho_R(M) \leq \mu_{\mathcal{K}}(M) \leq \overline{\sigma}(M) \tag{4}$$

In order to refine these bounds further we define the sets

$$\mathcal{Q}_{\mathcal{K}} = \{\Delta \in X_{\mathcal{K}} : \delta_i^r \in [-1\ 1], \delta_i^{c*}\delta_i^c = 1, \Delta_i^{C*}\Delta_i^C = I_{k_{m_r+m_c+i}}\} \tag{5}$$

$$\begin{aligned} \mathcal{D}_{\mathcal{K}} = \{block\ diag(&e^{j\theta_1} D_1, \ldots, e^{j\theta_{m_r}} D_{m_r}, D_{m_r+1}, \ldots, D_{m_r+m_c}, \\ &d_1 I_{k_{m_r+m_c+1}}, \ldots, d_{m_C} I_{k_m}) : \theta_i \in [-\frac{\pi}{2}\ \frac{\pi}{2}], \\ &0 < D_i = D_i^* \in \mathbf{C}^{k_i\times k_i}, 0 < d_i \in \mathbf{R}\} \end{aligned} \tag{6}$$

We note that these are not quite the "usual" scaling sets associated with μ, since for $m_r \neq 0$ matrices in $\mathcal{Q}_{\mathcal{K}}$ are not necessarily Unitary, and matrices in $\mathcal{D}_{\mathcal{K}}$ are not necessarily Hermitian. Nevertheless it still holds that for any $\Delta \in X_{\mathcal{K}}$ and any $D \in \mathcal{D}_{\mathcal{K}}$, $D\Delta = \Delta D$ and consequently we obtain

Lemma 1 *For any matrix $M \in \mathbf{C}^{n\times n}$, and any compatible block structure $\mathcal{K}$ then for all $D \in \mathcal{D}_{\mathcal{K}}$*

$$\mu_{\mathcal{K}}(M) = \mu_{\mathcal{K}}(DMD^{-1}) \tag{7}$$

Now in order to refine the lower bound we define the set

$$\mathbf{B}X_{\mathcal{K}} = \{\Delta \in X_{\mathcal{K}} : \overline{\sigma}(\Delta) \leq 1\} \tag{8}$$

Then the following lemma follows almost immediately from the definition of μ.

Lemma 2 *For any matrix $M \in \mathbf{C}^{n\times n}$, and any compatible block structure $\mathcal{K}$*

$$\mu_{\mathcal{K}}(M) = \max_{\Delta \in \mathbf{B}X_{\mathcal{K}}} \rho_R(\Delta M) \tag{9}$$

In the light of (7) and (9), noting that $\mathcal{Q}_{\mathcal{K}} \subset \mathbf{B}X_{\mathcal{K}}$ we can refine the bounds in (4) to obtain

Lemma 3 *For any matrix $M \in \mathbf{C}^{n\times n}$, and any compatible block structure $\mathcal{K}$*

$$\max_{Q\in\mathcal{Q}_{\mathcal{K}}} \rho_R(QM) \le \mu_{\mathcal{K}}(M) \le \inf_{D\in\mathcal{D}_{\mathcal{K}}} \overline{\sigma}(DMD^{-1}) \tag{10}$$

It is clear that the use of $\mathcal{D}_{\mathcal{K}}$ in the right hand inequality gains us nothing in terms of the upper bound over the more "usual" choice (see [4]) since any $D \in \mathcal{D}_{\mathcal{K}}$ may be factored into the product of a Unitary and a Hermitian matrix, and singular values are invariant under Unitary transformations. However we choose to work with this set for reasons which will become clear later.

3 Lower Bound as a Maximization

The lower bound (10) for the mixed case is a real eigenvalue maximization problem. In the purely complex case ($m_r = 0$) we can replace ρ_R by ρ and it was shown by Doyle [1] that in fact the lower bound is equal to μ. This represents a great reduction in the complexity of the problem over (9) since we are now only required to maximize over the boundary of the set. In this section we show that the lower bound for the mixed case (10) also holds with equality, and hence it is still sufficient to consider the complex uncertainties on their boundary. We note, however, that the definition of $\mathcal{Q}_{\mathcal{K}}$ requires us to search over the full range of the real perturbations. The following lemma is taken from [1].

Lemma 4 ([1]) *Let $p : \mathbf{C}^k \to \mathbf{C}$ be a (multivariable) polynomial and define $\beta = min\{\|z\|_\infty : p(z) = 0\}$ then there exists a $z \in \mathbf{C}^k$ such that $p(z) = 0$ and for every i, $|z_i| = \beta$*

This is now used to prove the main result of this section.

Theorem 1 *For any matrix $M \in \mathbf{C}^{n\times n}$, and any compatible block structure $\mathcal{K}$*

$$\max_{Q\in\mathcal{Q}_{\mathcal{K}}} \rho_R(QM) = \mu_{\mathcal{K}}(M) \tag{11}$$

Proof: Trivial from (10) if $\mu_{\mathcal{K}}(M) = 0$. So assume $\mu_{\mathcal{K}}(M) = \beta > 0$, and this value is achieved for some perturbation $\hat{\Delta}$, i.e. $det(I - \hat{\Delta}M) = 0$ and $\overline{\sigma}(\hat{\Delta}) \le \frac{1}{\beta}$. Now *fix* the real perturbations at these "optimal" values ($\delta_i^r = \hat{\delta}_i^r, i = 1,\ldots,m_r$ with $|\hat{\delta}_i^r| \le \frac{1}{\beta}$). Now allow the complex part of Δ to vary and consider minimizing $\overline{\sigma}(\Delta)$ subject to $det(I - \Delta M) = 0$. Performing an SVD on Δ we obtain $det(I - U\Sigma VM) = 0$ where U and V are (block diagonal) Unitary matrices and

$$\Sigma = diag(\hat{\delta}_1^r I_{k_1}, \ldots, \hat{\delta}_{m_r}^r I_{k_{m_r}}, \delta_1^c I_{k_{m_r+1}}, \ldots, \delta_{m_c}^c I_{k_{m_r+m_c}}, \gamma_1^c, \ldots, \gamma_k^c)$$

with $k = \sum_{i=m_r+m_c+1}^{m} k_i$. This is a polynomial in $\delta_1^c, \ldots, \delta_{m_c}^c, \gamma_1^c, \ldots, \gamma_k^c$ and so applying lemma 4 we have a solution with $|\hat{\delta}_1^c| = \ldots = |\hat{\delta}_{m_c}^c| = |\hat{\gamma}_1^c| = \ldots = |\hat{\gamma}_k^c| = \frac{1}{\hat{\beta}}$ and $\hat{\beta} \ge \beta$. Now suppose $\hat{\beta} > \beta$, say $\hat{\beta} = \beta + \epsilon$ for some $\epsilon > 0$, then since the roots of a polynomial are continuous functions of the coefficients we can find a $\delta > 0$ so that

$$|\delta_i^r - \hat{\delta}_i^r| < \delta, i = 1,\ldots,m_r \;\Rightarrow\; |\delta_i^c - \hat{\delta}_i^c| < \frac{\epsilon}{2}, i = 1,\ldots,m_c$$
$$|\gamma_i^c - \hat{\gamma}_i^c| < \frac{\epsilon}{2}, i = 1,\ldots,k$$

Then move each $|\delta_i^r|$ down by $\frac{\delta}{2}$ and we can find a Δ solving $det(I - \Delta M) = 0$ with $\overline{\sigma}(\Delta) < \frac{1}{\beta}$ contradicting the definition of μ. Thus $\hat{\beta} = \beta$ and it is now easy to check that for this solution $\beta\hat{\Delta} = \hat{Q} \in \mathcal{Q}_{\mathcal{K}}$ with $\rho_R(\hat{Q}M) = \beta = \mu_{\mathcal{K}}(M)$. □

4 Facts from Matrix Theory and Linear Algebra

4.1 Eigenvalue Perturbation Theory

This section reviews some eigenvalue perturbation theory we will need in Section 5. The material is fairly standard and is presented without proof (see [5] for further details).

Suppose we have a complex matrix $M(t) \in \mathbf{C}^{n\times n}$ depending analytically on the real parameter t. Then denote $M_0 := M(0)$ and suppose that this matrix has a distinct (i.e. algebraic multiplicity one) eigenvalue λ_0, with right and left eigenvectors x_0 and y_0 respectively, i.e. we have (after normalizing the eigenvectors appropriately)

$$\begin{aligned} y_0^* x_0 &= 1 \\ M_0 x_0 &= \lambda_0 x_0 \\ y_0^* M_0 &= \lambda_0 y_0^* \end{aligned}$$

Then for t in a sufficiently small neighborhood of the origin $M(t)$ has an eigenvalue $\lambda(t)$, with right and left eigenvectors $x(t)$ and $y(t)$ respectively, all of which depend analytically on t. i.e. we have

$$\begin{aligned} y(t)^* x(t) &= 1 \\ M(t)x(t) &= \lambda(t)x(t) \\ y^*(t)M(t) &= \lambda(t)y^*(t) \end{aligned}$$

with $\lambda(0) = \lambda_0$, $x(0) = x_0$, and $y(0) = y_0$. Thus we can differentiate this eigenvalue, with respect to t, and this yields

$$\dot{\lambda}(0) = y(0)^* \dot{M}(0) x(0) = y_0^* \dot{M}(0) x_0 \tag{12}$$

4.2 Linear Algebra Lemmas

The following two linear algebra lemmas are due to Packard [3].

Lemma 5 ([3]) *Let $y \in \mathbf{C}^n$ and $x \in \mathbf{C}^n$ be non-zero vectors. Then there exists a $d \in \mathbf{R}, d > 0$ such that $y = dx$ iff $Re(y^*Gx) \le 0$ for every $G \in \mathbf{C}^{n\times n}$ satisfying $G + G^* \le 0$.*

Lemma 6 ([3]) *Let $y \in \mathbf{C}^n$ and $x \in \mathbf{C}^n$ be non-zero vectors. Then there exists a Hermitian, Positive Definite $D \in \mathbf{C}^{n\times n}$ such that $y = Dx$ iff $y^*x \in \mathbf{R}$ and $y^*x > 0$.*

Now define the closed half space in the complex plane as, for some scalar $\psi \in \mathbf{R}$

$$H^\psi = \{z : Re(e^{-j\psi}z) \le 0\} \tag{13}$$

Then we have the following elementary linear algebra lemmas.

Lemma 7 *Given any set of complex scalars $\mathcal{Z} = \{z_i : i = 1,\ldots,m\}$ and any real scalar ψ. Then $\mathcal{Z} \subset H^\psi$ iff $\sum_{i=1}^m \alpha_i z_i \in H^\psi$ for all real non-negative scalars $\alpha_i, i = 1,\ldots,m$.*

Proof: ($\Leftarrow$) For each z_k choose $\alpha_k = 1$ and $\alpha_i = 0$ for $i \ne k$.

$$(\Rightarrow) \qquad Re\left(e^{-j\psi}\sum_{i=1}^m \alpha_i z_i\right)$$
$$= \sum_{i=1}^m \alpha_i Re\left(e^{-j\psi}z_i\right) \le 0 \qquad \square$$

Lemma 8 *Given any set of complex scalars $\mathcal{Z} = \{z_i : i = 1,\ldots,m\}$ define $\lambda := \sum_{i=1}^m \alpha_i z_i$ where $\alpha_i, i = 1,\ldots,m$ are real non-negative scalars. Then λ is not real and positive for any choice of the above α_i's iff $\mathcal{Z} \subset H^\psi$ for some $\psi \in (-\frac{\pi}{2}\ \frac{\pi}{2})$.*

Proof: ($\Leftarrow$) By lemma 7 $\mathcal{Z} \subset H^\psi$ implies $\lambda \in H^\psi$ and hence $Re(e^{-j\psi}\lambda) \le 0$. Suppose λ is real and positive. Then this implies $Re(e^{-j\psi}) \le 0$ which means $\psi \notin (-\frac{\pi}{2}\ \frac{\pi}{2})$ which is a contradiction.

($\Rightarrow$) Assume λ is never real and positive. Now suppose $\mathcal{Z} \not\subset H^\psi$ for any $\psi \in (-\frac{\pi}{2}\ \frac{\pi}{2})$. First choose $\psi = 0$. Then we must have at least one $z \in \mathcal{Z}$ with $Re(z) > 0$. Now we choose $\hat{z}_1$ as the element with $Re(z) > 0$ having minimum $|arg(z)|$ (which must be non-zero). Now choose $\psi = arg(\hat{z}_1)$. Then we must have a (non-zero) $\hat{z}_2 \in \mathcal{Z}$ with $\hat{z}_2 \notin H^\psi$. Suppose

$$\hat{z}_1 = r_1(\cos\psi + j\sin\psi), \qquad \hat{z}_2 = r_2(\cos\phi + j\sin\phi)$$

Then by our choice of $\hat{z}_1$ and $\hat{z}_2$ straightforward trigonometry yields the following facts: $|\sin\phi| \geq |\sin\psi|$, $sgn(\sin\phi) = -sgn(\sin\psi)$, $|\cos\phi| \leq |\cos\psi|$, and if $|\cos\phi| = |\cos\psi|$ then $\cos\phi = \cos\psi$. Now choose $\hat{\alpha}_1 = \frac{1}{r_1|\sin\psi|}$ and $\hat{\alpha}_2 = \frac{1}{r_2|\sin\phi|}$. Then we have

$$\hat{\lambda} = \hat{\alpha}_1\hat{z}_1 + \hat{\alpha}_2\hat{z}_2 = \frac{\cos\psi}{|\sin\psi|} + \frac{\cos\phi}{|\sin\phi|}$$

Thus $\hat{\lambda}$ is real and positive which is a contradiction. □

5 Characterization of a Maximum Point

We are interested in computing $\mu_{\mathcal{K}}(M)$, which by (9) and (11) is given by

$$\mu_{\mathcal{K}}(M) = \max_{\Delta\in\mathbf{B}X_{\mathcal{K}}} \rho_R(\Delta M) = \max_{Q\in\mathcal{Q}_{\mathcal{K}}} \rho_R(QM)$$

For reasons of tractability we choose to consider the problem $\max_{Q\in\mathcal{Q}_{\mathcal{K}}} \rho_R(QM)$. However since this is a non convex problem we will in general only be able to find local maxima, and hence we will obtain a lower bound for $\mu_{\mathcal{K}}(M)$ (which is the global maximum). We would like this lower bound be "tight" (i.e. close to μ) and so wish to rule out maxima of $\rho_R(QM)$ which we know are only local. Thus we only consider $Q \in \mathcal{Q}_{\mathcal{K}}$ which are local maxima of $\rho_R(QM)$ with respect not only to $Q \in \mathcal{Q}_{\mathcal{K}}$ but also to $Q \in \mathbf{B}X_{\mathcal{K}}$.

Note that for any $Q \in \mathcal{Q}_{\mathcal{K}}$ and any $\Delta \in \mathbf{B}X_{\mathcal{K}}$, then $Q\Delta \in \mathbf{B}X_{\mathcal{K}}$ and $\Delta Q \in \mathbf{B}X_{\mathcal{K}}$. Now suppose some matrix $Q \in \mathcal{Q}_{\mathcal{K}}$ achieves a local maximum of $\rho_R(QM)$ over $Q \in \mathbf{B}X_{\mathcal{K}}$. Then it is easy to show that the matrix $\hat{M} := QM$ has a local maximum of $\rho_R(\hat{Q}\hat{M})$ over $\hat{Q} \in \mathbf{B}X_{\mathcal{K}}$ at $\hat{Q} = I$. However since the real elements of Q are not restricted to be on their boundary we can say more than this. For any matrix $Q \in \mathcal{Q}_{\mathcal{K}}$ (see (5)) define the index sets

$$\mathcal{J}(Q) = \{i \leq m_r : |\delta_i^r| = 1\} \tag{14}$$

$$\hat{\mathcal{J}}(Q) = \{i \leq m_r : |\delta_i^r| < 1\} \tag{15}$$

and define the allowable perturbation set

$$\hat{\mathbf{B}}\Delta_\epsilon(\mathcal{J},\hat{\mathcal{J}}) = \{\Delta \in X_{\mathcal{K}} : |\delta_i^r| \leq 1, i \in \mathcal{J}, |\delta_i^r| < 1+\epsilon, i \in \hat{\mathcal{J}}, \\ |\delta_i^c| \leq 1, i = 1,\ldots,m_c, \overline{\sigma}(\Delta_i^C) \leq 1, i = 1,\ldots,m_C\} \tag{16}$$

We see that for sufficiently small $\epsilon > 0$ for any $Q \in \mathcal{Q}_{\mathcal{K}}$ and any $\Delta \in \hat{\mathbf{B}}\Delta_\epsilon(\mathcal{J}(Q),\hat{\mathcal{J}}(Q))$, then $Q\Delta \in \mathbf{B}X_{\mathcal{K}}$ and $\Delta Q \in \mathbf{B}X_{\mathcal{K}}$. The point of all this is that if some matrix $Q \in \mathcal{Q}_{\mathcal{K}}$ achieves a local maximum of $\rho_R(QM)$ over $Q \in \mathbf{B}X_{\mathcal{K}}$ then the matrix $\hat{M} := QM$ has a local maximum of $\rho_R(\hat{Q}\hat{M})$ *over* $\hat{Q} \in \hat{\mathbf{B}}\Delta_\epsilon(\mathcal{J}(Q),\hat{\mathcal{J}}(Q))$ (for some $\epsilon > 0$) at $\hat{Q} = I$ (and in fact the converse is true provided we assume that for every i, $\delta_i^r \neq 0$).

The notation here is unfortunately rather cumbersome and tends to obscure what is a really a rather simple concept. All that the above says is that if we are at a maximum point with some of the real perturbations at interior points (we do not need to consider this possibility for the complex perturbations) then we stay inside the allowable set, and cannot increase the function, if we move these *up* or down (in magnitude).

We introduce one further piece of notation. Suppose $M \in \mathbf{C}^{n\times n}$ has an eigenvalue λ with right and left eigenvectors x and y respectively. Then partition x and y compatibly with the block structure as

$$x = \begin{bmatrix} x_{r_1} \\ \vdots \\ x_{r_{m_r}} \\ x_{c_1} \\ \vdots \\ x_{c_{m_c}} \\ x_{C_1} \\ \vdots \\ x_{C_{m_C}} \end{bmatrix}, \qquad y = \begin{bmatrix} y_{r_1} \\ \vdots \\ y_{r_{m_r}} \\ y_{c_1} \\ \vdots \\ y_{c_{m_c}} \\ y_{C_1} \\ \vdots \\ y_{C_{m_C}} \end{bmatrix} \tag{17}$$

where $x_{r_i}, y_{r_i} \in \mathbf{C}^{k_i}$, $x_{c_i}, y_{c_i} \in \mathbf{C}^{k_{m_r+i}}$, $x_{C_i}, y_{C_i} \in \mathbf{C}^{k_{m_r+m_c+i}}$. These will be referred to as the "block components" of x and y, and we make a "non-degeneracy" assumption that for every i (in the appropriate set), $y_{r_i}^* x_{r_i} \neq 0, y_{c_i}^* x_{c_i} \neq 0, y_{C_i}^* x_{C_i} \neq 0$.

Theorem 2 *Suppose the matrix $M \in \mathbf{C}^{n\times n}$ has a distinct real eigenvalue $\lambda_0 > 0$ with right and left eigenvectors x and y respectively, satisfying the non-degeneracy assumption. Further suppose that $\rho_R(M) = \lambda_0$. Then if the function $\rho_R(QM)$ attains a local maximum over the set $Q \in \hat{\mathbf{B}}\Delta_\epsilon(\mathcal{J},\hat{\mathcal{J}})$ (for some $\epsilon > 0$) at $Q = I$ then there exists a matrix $D \in \mathcal{D}_{\mathcal{K}}$, with $\theta_i = \pm\frac{\pi}{2}$ for every $i \in \hat{\mathcal{J}}$, and a real scalar $\psi \in (-\frac{\pi}{2}\ \frac{\pi}{2})$, such that $y = e^{j\psi}Dx$.*

Proof: First we parametrize the perturbation set. Consider $G \in X_{\mathcal{K}}$ with

$$G = block\ diag(g_1^r I_{k_1},\ldots,g_{m_r}^r I_{k_{m_r}}, g_1^c I_{k_{m_r+1}},\ldots,g_{m_c}^c I_{k_{m_r+m_c}}, \\ G_1^C,\ldots,G_{m_C}^C) \tag{18}$$

and the added restrictions

$$\begin{aligned} g_i^r &\leq 0 \quad , \quad i \in \mathcal{J} \\ Re(g_i^c) &\leq 0 \quad , \quad i = 1,\ldots,m_c \\ G_i^C + G_i^{C*} &\leq 0 \quad , \quad i = 1,\ldots,m_C \end{aligned} \tag{19}$$

Now it can be shown that for some $\delta > 0$ then the set of all matrices $E(t) := (I+Gt)(I-Gt)^{-1}$ for $t \in [0\ \delta)$ and G as above is an open neighborhood of $\hat{\mathbf{B}}\Delta_\epsilon(\mathcal{J},\hat{\mathcal{J}})$ about $E(0) = I$. So now define the matrix $R(t) := E(t)M$. Then it is clear that $\rho_R(QM)$ has attained a local maximum over the set $Q \in \hat{\mathbf{B}}\Delta_\epsilon(\mathcal{J},\hat{\mathcal{J}})$ at $Q = I$ iff $\rho_R(R(t))$ has attained a local maximum over $t \in [0\ \delta)$ at $t = 0$ *for arbitrary G as above.*

Since $R(0) = M$ has a distinct real eigenvalue λ_0 we have (for some non-empty interval about the origin) an analytic function $\lambda(t)$, with $\lambda(0) = \lambda_0$, and $\lambda(t)$ an eigenvalue of $R(t)$. Thus we can differentiate to obtain

$$\dot{\lambda}(0) = y^*\dot{R}(0)x = 2y^*GMx = 2\lambda_0 y^*Gx \tag{20}$$

In block notation this becomes

$$\dot{\lambda}(0) = 2\lambda_0\left(\sum_{i=1}^{m_r} g_i^r y_i^{r*}x_i^r + \sum_{i=1}^{m_c} g_i^c y_i^{c*}x_i^c + \sum_{i=1}^{m_C} y_i^{C*}G_i^C x_i^C\right) \tag{21}$$

Define the set of points

$$\begin{aligned} \mathcal{Z} &= \{z_i : i = 1,\ldots,m\} = \{g_i^r y_i^{r*}x_i^r : i = 1,\ldots,m_r\} \cup \\ &\quad \{g_i^c y_i^{c*}x_i^c : i = 1,\ldots,m_c\} \cup \{y_i^{C*}G_i^C x_i^C : i = 1,\ldots,m_C\} \end{aligned} \tag{22}$$

with the obvious identification for the elements z_i. Now since we are at a maximum point we have that $\dot{\lambda}(0)$ is never real and positive. Thus, noting that we may independently scale g_i^r, g_i^c, G_i^C by arbitrary non-negative scalars and still satisfy (19), then applying lemma 8 to (21) and (22) gives that this is true iff $\mathcal{Z} \subset H^\psi$ for some $\psi \in (-\frac{\pi}{2}\ \frac{\pi}{2})$ for each $G \in X_{\mathcal{K}}$ satisfying (19). It is tedious but straightforward to verify that the following conditions are necessary and sufficient to ensure this (apply lemma 7 to (19) and use similar constructions to lemma 8):

$$\begin{aligned} Re(e^{j\psi}y_i^{r*}x_i^r) &\geq 0 \qquad i = 1,\ldots,m_r \\ Re(e^{j\psi}y_i^{r*}x_i^r) &= 0 \qquad i \in \hat{\mathcal{J}} \\ e^{j\psi}y_i^{c*}x_i^c &\in (0\ \infty) \qquad i = 1,\ldots,m_c \\ Re(e^{j\psi}y_i^{C*}G_i^C x_i^C) &\leq 0 \qquad \text{for all } G_i^C \text{ with } G_i^C + G_i^{C*} \leq 0, \\ &\qquad\qquad i = 1,\ldots,m_C \end{aligned} \tag{23}$$

for some $\psi \in (-\frac{\pi}{2}\ \frac{\pi}{2})$. Note that in the pure complex case the normalization condition $y^*x = 1$ implies $\psi = 0$. Since the scalar $e^{j\psi}$ terms may simply be absorbed into one of the vectors we can apply lemmas 5 and 6 to each block component of x and y to obtain the equivalent conditions

$$\begin{aligned} y_i^r &= e^{j\psi}e^{j\theta_i}D_i x_i^r \qquad 0 < D_i = D_i^*, \theta_i \in [-\frac{\pi}{2}\ \frac{\pi}{2}], i = 1,\ldots,m_r \\ y_i^r &= e^{j\psi}e^{j\theta_i}D_i x_i^r \qquad 0 < D_i = D_i^*, \theta_i = \pm\frac{\pi}{2}, i \in \hat{\mathcal{J}} \\ y_i^c &= e^{j\psi}D_i x_i^c \qquad 0 < D_i = D_i^*, i = 1,\ldots,m_c \\ y_i^C &= e^{j\psi}d_i x_i^C \qquad 0 < d_i \in \mathbf{R}, i = 1,\ldots,m_C \end{aligned} \tag{24}$$

Stacking these relations in matrix form yields $y = e^{j\psi}Dx$ with D of the required form. □

Remarks: We note from the proof that we immediately have a partial converse to theorem 2, namely that if $y = e^{j\psi}Dx$ under the above assumptions, then no directional derivative (in the above sense) of the eigenvalue achieving $\rho_R(QM)$ over the set $Q \in \hat{\mathbf{B}}\Delta_\epsilon(\mathcal{J}, \hat{\mathcal{J}})$ is real and positive at $Q = I$.

The assumption of a distinct eigenvalue was made to ensure differentiability. Since we are maximizing this eigenvalue it is generic that it will be distinct at the maximum, and hence this assumption should not prove restrictive. This also suggests that it should be possible to eliminate this assumption. Another assumption in the theorem statement is that of non-degeneracy, which will also hold generically. It appears that the degenerate cases where the proof breaks down correspond to saddle points. Thus it is conjectured that theorem 2 and all the results which follow hold with both of these assumptions removed, and this is being investigated.

6 A Decomposition at μ

Theorem 2 gives us a characterization of a maximum point of $\rho_R(QM)$ in terms of an alignment of the right and left eigenvectors of QM. This leads directly to the following decomposition.

Theorem 3 *Suppose $Q \in \mathcal{Q}_\mathcal{K}$ achieves the global maximum for the problem $\max_{Q\in\mathcal{Q}_\mathcal{K}} \rho_R(QM)$, and that the eigenvalue achieving $\rho_R(QM)$, denoted β, is distinct and positive. Then if the right and left eigenvectors of QM, denoted x and y respectively, satisfy the non-degeneracy assumption, there exists a matrix $D \in \mathcal{D}_\mathcal{K}$ with $D^2 \in \mathcal{D}_\mathcal{K}$ and $\theta_i = \pm\frac{\pi}{4}$ for $i \in \hat{\mathcal{J}}(Q)$ such that*

$$\begin{aligned} QDMD^{-1}(Dx) &= \beta Dx \\ (x^*D^*)QD^*M(D^*)^{-1} &= \beta x^*D^* \end{aligned} \tag{25}$$

with $\beta = \mu_\mathcal{K}(M)$.

Proof: By theorem 1 we immediately have $\beta = \mu_\mathcal{K}(M)$. Thus by lemma 2, $Q \in \mathcal{Q}_\mathcal{K}$ is also a global maximum of $\rho_R(QM)$ over $Q \in \mathbf{B}X_\mathcal{K}$. Hence the matrix $\hat{M} := QM$ achieves a maximum of $\rho_R(\hat{Q}\hat{M})$ over $\hat{Q} \in \hat{\mathbf{B}}\Delta_\epsilon(\mathcal{J}(Q), \hat{\mathcal{J}}(Q))$ (for some $\epsilon > 0$) at $\hat{Q} = I$. Now apply theorem 2 to conclude $y = e^{j\psi}\hat{D}x$ with $\hat{D} \in \mathcal{D}_\mathcal{K}$ and $\hat{\theta}_i = \pm\frac{\pi}{2}$ for $i \in \hat{\mathcal{J}}(Q)$. Now define D as the unique matrix such that $D \in \mathcal{D}_\mathcal{K}$ and $D^2 = \hat{D}$. Substitution of this into the right and left eigenvalue equations of QM and simple manipulations (note that for any $Q \in \mathcal{Q}_\mathcal{K}$ and any $D \in \mathcal{D}_\mathcal{K}$, Q and D commute) yields the result. □

Remarks: We note from the proof that we have a decomposition as above (but with β not necessarily equal to $\mu_\mathcal{K}(M)$) at any $Q \in \mathcal{Q}_\mathcal{K}$ which maximizes $\rho_R(QM)$ over $Q \in \mathbf{B}X_\mathcal{K}$ under the above assumptions on the eigenvalue and eigenvectors.

Employing simple manipulations of (25) yields a partial converse of this theorem. If we have a decomposition as in (25) with β real and positive and x non-zero, then we have that β is an eigenvalue of QM with right and left eigenvectors x and y respectively (thus β is a lower bound for $\mu_\mathcal{K}(M)$) where $y = re^{j\psi}D^2x$ with D as above, r a positive real scalar (which we could thus absorb into D), and $\psi \in [-\frac{\pi}{2}\ \frac{\pi}{2}]$. If we add the further technical assumption that we are not in the special case of $\theta_i = \pm\frac{\pi}{4}$ for all $i = 1, \ldots, m_r$ and $m_c = 0, m_C = 0$ then we have $\psi \in (-\frac{\pi}{2}\ \frac{\pi}{2})$.

It is well known that for the purely complex case we have a decomposition at μ (see Packard [3]) and (25) extends this result to the mixed case ($m_r \neq 0$).

Thus we (almost) always have a decomposition at μ of the form (25), and any such decomposition gives us a lower bound for μ. Now we reformulate this condition into a set of vector equations.

Lemma 9 *Suppose we have matrices $Q \in \mathcal{Q}_\mathcal{K}$ with $\delta_i^r \neq 0$ for $i = 1, \ldots, m_r$ and $\hat{D} \in \mathcal{D}_\mathcal{K}$ with $\hat{D}^2 \in \mathcal{D}_\mathcal{K}$ and $\hat{\theta}_i = \pm\frac{\pi}{4}$ for $i \in \hat{\mathcal{J}}(Q)$. Then we have a non-zero vector $\hat{x}$, and a real positive scalar β such that*

$$\begin{aligned} Q\hat{D}M\hat{D}^{-1}(\hat{D}\hat{x}) &= \beta\hat{D}\hat{x} \\ (\hat{x}^*\hat{D}^*)Q\hat{D}^*M(\hat{D}^*)^{-1} &= \beta\hat{x}^*\hat{D}^* \end{aligned} \tag{26}$$

iff there exists a matrix $D \in \mathcal{D}_\mathcal{K}$ with $\theta_i = \pm\frac{\pi}{2}$ for $i \in \hat{\mathcal{J}}(Q)$ and non-zero vectors b, a, z, w such that

$$\begin{aligned} Mb = \beta a \quad & M^*z = \beta w \\ b = Qa \quad & b = D^{-1}w \\ z = Q^*QDa \quad & z = Q^*w \end{aligned} \tag{27}$$

Proof: ($\Rightarrow$) Define $x = \hat{D}\hat{x}$ and b, a, z, w as

$$\begin{aligned} b &= \hat{D}^{-1}x & a &= \hat{D}^{-1}Q^{-1}x \\ z &= \hat{D}Q^*x & w &= \hat{D}x \end{aligned}$$

Finally define $D = \hat{D}^2$ and the result follows.

($\Leftarrow$) Defining $\hat{D}$ as the unique matrix $\hat{D} \in \mathcal{D}_\mathcal{K}$ such that $\hat{D}^2 = D$, and $\hat{x} = b$ the result follows directly. □

Remarks: We note that assumption $\delta_i^r \neq 0$ for $i = 1, \ldots, m_r$ was included to ensure that Q was non-singular. This assumption was used in showing the necessity of (27) but was not required to show sufficiency of (27).

7 A Power Algorithm for the Lower Bound

In light of lemma 9 the problem of computing a lower bound for $\mu_\mathcal{K}(M)$ is reduced to one of finding a solution to the set of equations in (27) which gives us a decomposition as in (25). In order to do this we first note that if we partition b, a, z, w compatibly with the block structure as in (17) then the set of constraint equations

$$\begin{aligned} b = Qa \quad & b = D^{-1}w \\ z = Q^*QDa \quad & z = Q^*w \end{aligned}$$

can be broken down into a series of m similar independent constraint equations on the block components (since Q and D are block diagonal). These equations are of three types corresponding to a repeated real scalar block, a repeated complex scalar block, or a full complex block. We now consider a generic constraint of each type. The following two lemmas are due to Packard [3].

Lemma 10 (Repeated Complex Scalar Block [3]) *Let $b, a, z, w \in \mathbf{C}^k$ be non-zero vectors with $a^*w \neq 0$. Then there exists a complex scalar q with $|q| = 1$, and a complex matrix $D \in \mathbf{C}^{k\times k}$ with $0 < D = D^*$ such that*

$$\begin{aligned} b = qa \quad & b = D^{-1}w \\ z = q^*qDa \quad & z = q^*w \end{aligned}$$

if and only if

$$z = \frac{w^*a}{|w^*a|}w \qquad b = \frac{a^*w}{|a^*w|}a \tag{28}$$

Lemma 11 (Full Complex Block [3]) *Let $b, a, z, w \in \mathbf{C}^k$ be non-zero vectors. Then there exists a complex matrix $Q \in \mathbf{C}^{k\times k}$ with $Q^*Q = I_k$, and a real positive scalar d such that*

$$\begin{aligned} b = Qa \quad & b = d^{-1}w \\ z = Q^*Qda \quad & z = Q^*w \end{aligned}$$

if and only if

$$z = \frac{|w|}{|a|}a \qquad b = \frac{|a|}{|w|}w \tag{29}$$

Now we consider a repeated real scalar block, bearing in mind that we have additional constraints if the real perturbation is not on the boundary (i.e. for $i \in \hat{\mathcal{J}}(Q)$)

Lemma 12 (Repeated Real Scalar Block) *Let $b, a, z, w \in \mathbf{C}^k$ be non-zero vectors with $a^*w \neq 0$. Then we have a real scalar q with $|q| \leq 1$, a real scalar $\theta \in [-\frac{\pi}{2}\ \frac{\pi}{2}]$, and a complex matrix $D \in \mathbf{C}^{k\times k}$ with $0 < D = D^*$ such that*

$$\begin{array}{ll} b = qa & b = e^{-j\theta}D^{-1}w \\ z = q^*qe^{j\theta}Da & z = q^*w \end{array}$$

with $\theta = \pm\frac{\pi}{2}$ for $|q| < 1$ iff

$$z = qw \qquad b = qa \tag{30}$$

with

$$\begin{array}{lll} Re(a^*w) \geq 0 & for & q = 1 \\ Re(a^*w) \leq 0 & for & q = -1 \\ Re(a^*w) = 0 & for & |q| < 1 \end{array} \tag{31}$$

Proof: ($\Rightarrow$) Immediately we have $z = qw$ and $b = qa$. Thus $a^*w = \frac{1}{q}b^*w = \frac{1}{q}e^{j\theta}w^*(D^*)^{-1}w$. Now $q = 1$ implies $arg(a^*w) = \theta$ and hence $Re(w^*a) \geq 0$. Similarly $q = -1$ implies $arg(a^*w) = \theta + \pi$ and hence $Re(w^*a) \leq 0$. Finally $|q| < 1$ implies $arg(a^*w) = \theta$ or $\theta + \pi$ with $\theta = \pm\frac{\pi}{2}$. Thus $arg(a^*w) = \pm\frac{\pi}{2}$ and so $Re(a^*w) = 0$.

($\Leftarrow$) Immediately we have $b = qa$ and $z = q^*w$, and so $b^*w = qa^*w$. Denoting $\theta = arg(b^*w)$ we see that for $q = 1$ $Re(a^*w) \geq 0$ which implies $Re(b^*w) \geq 0$ and so $\theta \in [-\frac{\pi}{2}\ \frac{\pi}{2}]$. Similarly for $q = -1$ $Re(a^*w) \leq 0$ which implies $Re(b^*w) \geq 0$ and so $\theta \in [-\frac{\pi}{2}\ \frac{\pi}{2}]$. Finally for $|q| < 1$ $Re(a^*w) = 0$ which implies $Re(b^*w) = 0$ and so $\theta = \pm\frac{\pi}{2}$. Now $b^*(e^{-j\theta}w)$ is real and positive and so applying lemma 6 we have a matrix $\hat{D}$ with $0 < \hat{D} = \hat{D}^*$ such that $b = e^{-j\theta}\hat{D}w$. Define $D = \hat{D}^{-1}$ and we have $b = e^{-j\theta}D^{-1}w$ and $z = q^*w = q^*e^{j\theta}Db = q^*qe^{j\theta}Da$. □

These lemmas now allow us (with a few technical assumptions) to eliminate the matrices Q and D from (27). In order to avoid the notation becoming excessive we consider a simple block structure with $m_r = m_c = m_C = 1$ for the remainder of this section. *We stress that this is purely for notational convenience, and that the general formulae for an arbitrary block structure, as defined in Section 2, are simply obtained by duplicating the appropriate formulae for each block.* So given $\mathcal{K} = (k_1, k_2, k_3)$the appropriate scaling sets become

$$\begin{aligned} \mathcal{Q}_{sub} = \{&block\ diag(q^rI_{k_1}, q^cI_{k_2}, Q^C) : q^r \in [-1\ 1], \\ &q^{c*}q^c = 1, Q^{C*}Q^C = I_{k_3}\} \end{aligned} \tag{32}$$

$$\begin{aligned} \mathcal{D}_{sub} = \{&block\ diag(e^{j\theta}D_1, D_2, d_1I_{k_3}) : \theta \in [-\frac{\pi}{2}\ \frac{\pi}{2}], \\ &0 < D_i = D_i^* \in \mathbf{C}^{k_i\times k_i}, 0 < d_1 \in \mathbf{R}\} \end{aligned} \tag{33}$$

and we partition b, a, z, w compatibly with this block structure as

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \quad w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \tag{34}$$

where $b_i, a_i, z_i, w_i \in \mathbf{C}^{k_i}$. Then we obtain our final form of (27) as

Theorem 4 *Suppose we have vectors $b, a, z, w \in \mathbf{C}^n$ partitioned as in (34) with $b_i, a_i, z_i, w_i \neq 0$ and $a_1^*w_1, a_2^*w_2 \neq 0$. Then there exist matrices $Q \in \mathcal{Q}_{sub}$ and $D \in \mathcal{D}_{sub}$, and a positive real scalar β such that*

$$\begin{array}{ll} Mb = \beta a & M^*z = \beta w \\ b = Qa & b = D^{-1}w \\ z = Q^*QDa & z = Q^*w \end{array}$$

with $\theta \in [-\frac{\pi}{2}\ \frac{\pi}{2}]$ and $\theta = \pm\frac{\pi}{2}$ for $|q^r| < 1$ iff

$$\begin{aligned} &Mb = \beta a \\ &z_1 = qw_1 \qquad z_2 = \frac{w_2^*a_2}{|w_2^*a_2|}w_2 \qquad z_3 = \frac{|w_3|}{|a_3|}a_3 \\ &M^*z = \beta w \\ &b_1 = qa_1 \qquad b_2 = \frac{a_2^*w_2}{|a_2^*w_2|}a_2 \qquad b_3 = \frac{|a_3|}{|w_3|}w_3 \end{aligned} \tag{35}$$

for some real scalar $q \in [-1\ 1]$ with

$$\begin{array}{lll} Re(a_1^*w_1) \geq 0 & for & q = 1 \\ Re(a_1^*w_1) \leq 0 & for & q = -1 \\ Re(a_1^*w_1) = 0 & for & |q| < 1 \end{array} \tag{36}$$

Proof: Apply lemmas 10, 11 and 12 to the appropriate block components. □

Remarks: Since the relationships (35) and (36) are unaffected if we multiply b and a by an arbitrary positive real scalar α, and z and w by an arbitrary positive real scalar γ, then in searching for solutions to these equations we may impose the additional restriction $|a| = |w| = 1$.

Any solution to (35) and (36) immediately gives us a decomposition as in (25) and hence β is a lower bound for $\mu_{\mathcal{K}}(M)$. We also note that, under certain technical assumptions (as given), there always exists a solution to these equations with $\beta = \mu_{\mathcal{K}}(M)$. We now propose finding a solution to this system of equations via the following power iteration:

$$\begin{aligned} \tilde{\beta}_{k+1}a_{k+1} &= Mb_k \\ z_{1_{k+1}} &= \tilde{q}_{k+1}w_{1_k} \quad z_{2_{k+1}} = \frac{w_{2_k}^*a_{2_{k+1}}}{|w_{2_k}^*a_{2_{k+1}}|}w_{2_k} \\ z_{3_{k+1}} &= \frac{|w_{3_k}|}{|a_{3_{k+1}}|}a_{3_{k+1}} \\ \hat{\beta}_{k+1}w_{k+1} &= M^*z_{k+1} \\ b_{1_{k+1}} &= \hat{q}_{k+1}a_{1_{k+1}} \quad b_{2_{k+1}} = \frac{a_{2_{k+1}}^*w_{2_{k+1}}}{|a_{2_{k+1}}^*w_{2_{k+1}}|}a_{2_{k+1}} \\ b_{3_{k+1}} &= \frac{|a_{3_{k+1}}|}{|w_{3_{k+1}}|}w_{3_{k+1}} \end{aligned} \tag{37}$$

where $\tilde{q}_{k+1}$ and $\hat{q}_{k+1}$ evolve as

$$\begin{aligned} \tilde{\alpha}_{k+1} &= sgn(\hat{q}_k)\frac{|b_{1_k}|}{|a_{1_{k+1}}|} + Re(a_{1_{k+1}}^*w_{1_k}) \\ &\text{If } |\tilde{\alpha}_{k+1}| \geq 1 \text{ then } \tilde{q}_{k+1} = \frac{\tilde{\alpha}_{k+1}}{|\tilde{\alpha}_{k+1}|} \\ &\text{Else } \tilde{q}_{k+1} = \tilde{\alpha}_{k+1} \\ \hat{\alpha}_{k+1} &= sgn(\tilde{q}_{k+1})\frac{|b_{1_k}|}{|a_{1_{k+1}}|} + Re(a_{1_{k+1}}^*w_{1_{k+1}}) \\ &\text{If } |\hat{\alpha}_{k+1}| \geq 1 \text{ then } \hat{q}_{k+1} = \frac{\hat{\alpha}_{k+1}}{|\hat{\alpha}_{k+1}|} \\ &\text{Else } \hat{q}_{k+1} = \hat{\alpha}_{k+1} \end{aligned} \tag{38}$$

and $\tilde{\beta}_{k+1}, \hat{\beta}_{k+1}$ are chosen positive real so that $|a_{k+1}| = |w_{k+1}| = 1$.

It is now straightforward to verify that if the algorithm converges to some equilibrium point then we satisfy the appropriate constraints on each block component and hence by lemmas 10, 11, and 12 we have have non-zero vectors $b, a, z, w \in \mathbf{C}^n$, matrices $Q \in \mathcal{Q}_{sub}, D \in \mathcal{D}_{sub}$, and positive real scalars $\tilde{\beta}, \hat{\beta}$ such that

$$\begin{array}{ll} Mb = \tilde{\beta}a & M^*z = \hat{\beta}w \\ b = Qa & b = D^{-1}w \\ z = Q^*QDa & z = Q^*w \end{array} \tag{39}$$

Thus if $\tilde{\beta} = \hat{\beta}$ then we satisfy (27) and so have a decomposition as in (25), and hence $\tilde{\beta}$ is a lower bound for $\mu_{\mathcal{K}}(M)$. We note that if $\tilde{\beta} \neq \hat{\beta}$

then we have not found a decomposition as in (25), however we still have $QMb = \tilde{\beta}b$ and $w^*QM = \hat{\beta}w^*$, and so $\max(\tilde{\beta}, \hat{\beta})$ still gives us a lower bound for $\mu_{\mathcal{K}}(M)$.

Note that the equilibrium points of the algorithm are unaffected if we multiply the terms $Re(a^*_{1_{k+1}} w_{1_k})$, $Re(a^*_{1_{k+1}} w_{1_{k+1}})$ by arbitrary real positive scalars, and hence we may employ this degree of freedom to select scaling parameters so as to aid convergence.

For the purely complex case, $m_r = 0$, this algorithm reduces to that of Packard [3], and hence many of the comments made there also apply here. In particular we note that there is a potential problem with the algorithm if any of the following occur:

- $Mb_k = 0$ or $M^* z_k = 0$
- $|a_{1_k}| = 0$
- $|a^*_{2_k} w_{2_k}| = 0$
- $|a_{3_k}| = 0$ or $|w_{3_k}| = 0$

since any of these conditions will result in one of the terms in the algorithm being undefined. If this occurs then simply restart the algorithm from a new point (i.e. a new b_1, w_1 and $\tilde{\alpha}_2$). A scheme for computing the initial guesses b_1 and w_1 for Packard's algorithm is outlined in [3] and it would also seem to be a cheap way to generate reasonable first guesses here. The first guess for $\tilde{\alpha}_2$ is then simply chosen as $|\tilde{\alpha}_2| = \frac{|b_{1_1}|}{|a_{1_2}|}$ with $sgn(\tilde{\alpha}_2)$ chosen so as to minimize $|b_{1_1} - \tilde{\alpha}_2 a_{1_2}|$.

8 Numerical Experience

The algorithm outlined in Section 7 has been implemented and initial test results are encouraging. The potential problems of certain terms becoming undefined do not seem to occur in practice (although it is possible to construct matrices for which this occurs). It also appears that we obtain $\tilde{\beta} = \hat{\beta}$ in practice, and hence the algorithm gives us not only a lower bound for $\mu_{\mathcal{K}}(M)$ but also a decomposition as in (25). Both of these issues are subjects of current research.

For the purpose of comparison we used the upper bound of Fan et. al. [4]. Whilst numerical experience is still somewhat limited (due in part to our current lack of an efficient implementation of this upper bound) the lower bound algorithm has been tested on a fairly large number of random matrices. For a typical test run of 500 randomly distributed 5×5 complex matrices with 2 real scalar uncertainties and three complex scalar uncertainties the algorithm converged 96% of the time, in an average of 22 iterations, with an average ratio of 0.96 to the upper bound.

The algorithm has also been tested on a variety of other block structures, and on much larger matrices (e.g. 100×100) and the convergence properties appear similar to those described above. An exception to this is the case $m_c = m_C = 0$ which appears to have significantly poorer convergence properties than any other. There are important reasons for this that seem inherent to the problem, not the computation scheme. This will be the subject of further papers. Whilst the results in this paper do apply to this case, the real-only case is of less engineering interest than the mixed case, since any robust performance μ test will always involve at least one complex block.

For mixed problems, the growth rate in computational cost as a function of problem size appears to be reasonable. Experiments were carried out on random $n \times n$ complex matrices with $n = 10, 20, 50, 100$, with block structures consisting of all scalar blocks with $n/2$ real and $n/2$ complex. The growth rate in computation time for the existing implementation was less than n, but this is probably an artifice of the implementation in MATLAB, an interprative language. A more realistic measure is in terms of total floating point operations (flops), where the growth rate is approximately n^2. This type of growth rate is similar to that of power iterations for eigenvalues and singular values.

The convergence properties of standard eigenvalue and singular value power algorithms can be improved by inverse iteration, and similar adaptations to the algorithm described in (37) and (38) have been investigated. While the results are very preliminary, it appears that the convergence can be improved (to a success rate of better than 99% for the above example for instance). We believe further improvements are possible and this is being investigated.

9 Conclusion

The computation of a lower bound for μ has been shown to be equivalent to finding a certain matrix decomposition, and this in turn has been shown to be equivalent to finding a solution to a set of matrix-vector equations. This representation naturally leads to a power iteration scheme to generate a lower bound for μ. This scheme has been found to have fairly good convergence properties, and work is under way to improve this further. Each iteration of the scheme is very cheap, requiring only such operations as matrix-vector multiplications and vector inner products, and the method is sufficiently general to handle arbitrary numbers of repeated real scalars, repeated complex scalars, and full complex blocks.

To reliably compute μ, the lower bound described in this paper would need to be combined with an upper bound. The upper bound of Fan, et. al. [4] appears promising as it involves convex optimization very similar to existing upper bounds for the purely complex case. For most problems, these bounds will be close enough, but there are matrices for which the gap can be large and the bounds will need to be refined. A promising approach is to use a standard branch and bound scheme such as the ones suggested by deGaston, Safonov, Sideris, et. al. (see [6] and references therein). The experimental work in [6] suggests that using their bounds, the growth rate of the subdomains in the branch and bound was modest and the total cost of computation is dominated by the cost of computing the bounds. Whether similar properties will hold using the bounds in this paper is not clear, as the partitioning algorithm in [6] made explicit use of information obtained in computing the bounds. Nevertheless, the results are encouraging and this avenue will be investigated.

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