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Fundamental Limitations of Disturbance Attenuation in the Presence of Side Information

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Abstract— **In this paper, we study fundamental limitations of disturbance attenuation of feedback systems, under the assumption that the controller has a finite horizon preview of the disturbance. In contrast with prior work, we extend Bode's integral equation for the case where the preview is made available to the controller via a general, finite capacity, communication system. Under asymptotic stationarity assumptions, our results show that the new fundamental limitation differs from Bode's only by a constant, which quantifies the information rate through the communication system. In the absence of stationarity, we derive a universal lower bound which uses entropy rates as a measure of performance.**

I. INTRODUCTION

Since it was first published in 1945 [1], Bode's integral equation is one of the most significant results in the theory of linear feedback. If $S(z)$ is the sensitivity transfer function [5] of a single-input linear feedback loop, in discrete time, then Bode's integral equation can be written as:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log(|S(e^{j\omega})|)d\omega = \sum_{\lambda\in\mathcal{UP}}\log|\lambda| \qquad (1)$$

where $\mathcal{UP}$ are the unstable poles of the open loop system [5], which is assumed to be rational and strictly proper. By using feedback, one would expect that disturbance rejection can be improved. On the other hand, (1) quantifies a fundamental limitation which says that disturbance rejection can be, at most, *shaped* in frequency. Equivalently, $|S(e^{j\omega})|$ cannot be made small at all frequencies. Due to its importance, Bode's fundamental limitation has been extended to more general frameworks [23] than the linear and time invariant one. The multi-dimensional version was provided in [9], while the time-varying case has been addressed in [13] and certain non-linear systems have been analyzed in [29], [12], [24]. In recent publications, such as [16] and references therein, the study of fundamental limitations generalizes to controllers with preview.

Using an information theoretic formulation, Bode's result was extended for feedback systems where the controller belongs to a general class [14], [15], which might include systems operating on a discrete or finite alphabet. The use of information theoretic quantities, which was first suggested in [12], also allows for the clear differentiation of the roles of causality and information flow in the feedback loop [14], [15]. While Causality is responsible for Bode's fundamental limitation, information constraints in the feedback loop give rise to a new limitation[14], [15]. The work in [6] also explores the connection between Bode's integral formula and the ability to transmit information over a Gaussian channel, by means of linear and time invariant systems acting as encoders and decoders. Bode's fundamental limitation is derived for a deterministic setting in [26], under certain convergence conditions.

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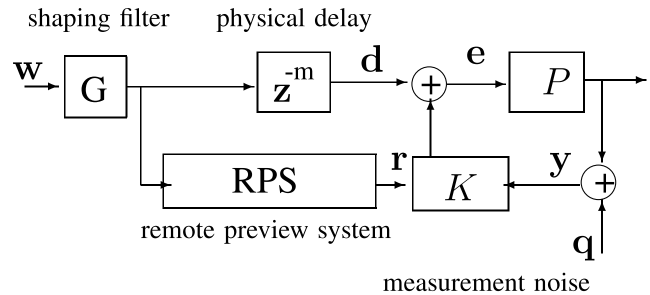

Fig. 1. Structure of a Remote Preview System.

A. Main Contributions

It is well known that the use of disturbance previews may improve controller performance [28]. In [19] one finds recent results in optimal preview control as well as a source of references to other related approaches. Recent results on fundamental limitations in the presence of reference preview are given in [2], [16].

In this publication, we consider the diagram of Fig 1, where the controller has access to a remotely transmitted disturbance preview, represented as $\mathbf{r}$. This scheme portrays a formulation where the disturbance results from a physical phenomenon, which must travel in space until it reaches the system. The travel time is represented as a delay of m units of time. At the same time, a remote preview signal $\mathbf{r}$ may be available to the controller, subject to information transmission/processing constraints at the remote preview system (RPS) block. We also adopt a Markovian model for the disturbance, where G is an auto-regressive shaping filter and $\mathbf{w}$ is the innovations process.

This work characterizes the fundamental limits of preview control in a general remote setting. Examples of remote preview systems can be found in animal life, such as the ones that use vision and hearing to perceive a future physical interaction. In these cases, the information/processing constraints arise from limited vision and hearing resolution as well as noise and limited information processing in the brain [8]. Another example can be found in the information path of a heat-shock mechanism at the cellular level [7]. Further examples can be found in navigation engineering systems.

There are two extreme cases in this setup: The first is when the disturbance can be fully transmitted[1], in that case the disturbance can be canceled by the controller, and the second is the absence of remote preview information because that is the classical framework.

In this paper, we study the situation in between, i.e., we consider that $C > I_\infty(\mathbf{r},\mathbf{d}) > 0$, where C is a finite positive constant, which represents the Shannon capacity [3] of the RPS block, and $I_\infty(\mathbf{r},\mathbf{d})$ is the mutual information rate[2] between the disturbance $\mathbf{d}$ and the remote preview signal $\mathbf{r}$, in bits per unit of time.

The following summarizes the contributions of this paper:

- we consider a new type of networked control system, where a preview of the disturbance is available to the controller via a general communication system. We derive an extension of Bode's integral formula for the aforementioned scheme.
- by making use of information theoretic principles, our fundamental limitations incorporate explicitly the information rate constraints of the remote preview communication system.
- our derivations are valid for any strictly causal feedback loop, which includes time-varying, non-linear controllers operating over arbitrary alphabets.

B. Main results and paper organization

The main result in the paper is stated in Theorem 3.1. Under asymptotic stationary assumptions, such result has the following frequency domain interpretation, derived in theorem 3.4:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log[S(\omega)]d\omega \geq \sum_{i=1}^{n}\max\{0,\log(|\lambda_i(A)|)\} - C \quad (2)$$

where $S(\omega)$ is an appropriate generalization of the sensitivity function and A is the dynamic matrix of the plant P. In comparison with (1), the inequality (2) differs on the right-hand side by C, the upper-bound on $I_\infty(\mathbf{r},\mathbf{d})$. It shows that the fundamental limitation, in terms of $\log[S(\omega)]$, is similar to Bode's formula.

The paper is organized as follows: Sections I-C and I-D introduce the notation and main definitions. The technical framework is given in section II, where we also describe the measures of performance adopted in the paper. The main results are stated in section III. The proof of the main theorem as well as auxiliary results, which require intense use of information theory, can be found in section IV.

C. Notation

The following notation is adopted:

- Whenever it is clear from the context, we refer to a sequence $\{a(k)\}_0^\infty$ of elements in $\mathbb{R}^n$ as a. A finite segment of a sequence a is indicated as $a_{k_{min}}^{k_{max}} \overset{def}{=} \{a(k)\}_{k_{min}}^{k_{max}}$. If $k_{max} < k_{min}$ then $a_{k_{min}}^{k_{max}} = \emptyset$.
- Random variables are represented using boldface letters, such as $\mathbf{a}$.
- If $\mathbf{a}(k)$ is a stochastic process, then we use $a(k)$ to indicate a specific realization. Similar to the convention used for sequences, we may denote $\{\mathbf{a}(k)\}_0^\infty$ just as $\mathbf{a}$ and $\{a(k)\}_0^\infty$ as a. A finite segment of a stochastic process is indicated as $\mathbf{a}_{k_{min}}^{k_{max}}$.
- The probability density of a random variable $\mathbf{a}$, if it exists, is denoted as p_a. The conditional probability, given $\mathbf{b}$, is indicated as $p_{a|b}$.
- The expectation operator over $\mathbf{a}$ is written as $\mathcal{E}[\mathbf{a}]$.
- We write $\log_2(.)$ simply as $\log(.)$.
- We adopt the convention $0\log 0 = 0$.

D. Basic Definitions of Information Theory

In this section, we summarize the main definitions of Information Theory which are used throughout the paper. We adopt [21], as a primary reference, because it contemplates general probabilistic spaces in a unified framework. We define mutual information, between any two random variables, as:

Definition 1.1: (from [21] pp. 9) The mutual information $I : (\mathbf{a};\mathbf{b}) \rightarrow \mathbf{R}_+ \bigcup\{\infty\}$, between $\mathbf{a}$ and $\mathbf{b}$, is given by $I(\mathbf{a};\mathbf{b}) = \sup\sum_{ij}\mathcal{P}_{\mathbf{a},\mathbf{b}}(E_i\times F_j)\log\frac{\mathcal{P}_{\mathbf{a},\mathbf{b}}(E_i\times F_j)}{\mathcal{P}_{\mathbf{a}}(E_i)\mathcal{P}_{\mathbf{b}}(F_j)}$, where the supremum is taken over all partitions $\{E_i\}$ of $\mathcal{A}$ and $\{F_j\}$ of $\mathcal{B}$.
The definition of conditional mutual information can be found in [21] (pp. 37).

Notice that, in definition 1.1, $\mathcal{A}$ and $\mathcal{B}$ may be different. Without loss of generality, we consider probability spaces which are countable or $\mathbb{R}^q$, for some q. We also define the following quantities, denoted as differential entropy and conditional differential entropy, which are useful in the computation of $I(\cdot,\cdot)$ for certain cases relevant in this paper.

Definition 1.2: If $\mathbf{a}$ is a random variable with alphabet $\mathcal{A} = \mathbb{R}^q$, finite covariance matrix Σ_a and a bounded[3] and measurable probability density function $p_a(\cdot)$ then we define the differential entropy of $\mathbf{a}$ as $h(\mathbf{a}) = -\int_{\mathbb{R}^q} p_a(\gamma)\log p_a(\gamma)d\gamma$. If $\mathbf{b}$ is a random variable with alphabet $\mathcal{B} = \mathbb{R}^{q'}$ and such that $p_{a,b}(\cdot,\cdot)$ is a bounded measurable probability density function, with finite covariance, then we define the conditional differential entropy of $\mathbf{a}$ given $\mathbf{b}$ as[4]:

$$h(\mathbf{a}|\mathbf{b}) = h(\mathbf{a},\mathbf{b}) - h(\mathbf{b}) = \\ -\int_{\mathbb{R}^{q'}}\int_{\mathbb{R}^q} p_{a,b}(\gamma_a,\gamma_b)\log p_{a|b}(\gamma_a,\gamma_b)d\gamma_a d\gamma_b \quad (3)$$

If $\mathcal{B}$ is countable and $p_{a|b}(\gamma_a, b)$ is bounded and $\log p_{a|b}$ is measurable in the measure induced in $\mathcal{A}\times\mathcal{B}$ then $h(\mathbf{a}|\mathbf{b})$

[1]This would require a RPS block with infinite capacity.

[2]This quantity is precisely defined in section I-D.

[3]Since p_a is bounded with a finite covariance matrix Σ_a it follows that $h(\mathbf{a}) < \infty$. The fact that $h(\mathbf{a}) < \infty$ further implies that $p_a \log p_a$ is integrable. Proofs of these facts use standard analysis arguments and can be found in [15]

[4]Notice that the equalities bellow hold because all the integrands are integrable

is defined as:

$$h(\mathbf{a}|\mathbf{b}) = -\sum_{\gamma_b \in \mathcal{B}} \int_{\mathbb{R}^q} p_{a,b}(\gamma_a,\gamma_b) \log p_{a|b}(\gamma_a,\gamma_b) d\gamma_a \quad (4)$$

Likewise, the quantity $h(\mathbf{a}|\mathbf{b},\mathbf{c})$ is defined by incorporating another sum if the alphabet of $\mathbf{c}$ is discrete, or an integral otherwise. Notice that the quantity defined in (4) may not be bounded, because the integrand is not necessarily integrable/summable. In the more general case, if we write $h(\mathbf{a}|\mathbf{b})$ then we assume that $p_{a|b}$ is bounded and that $\log p_{a|b}$ is integrable with respect to the probability measure induced in $\mathcal{A} \times \mathcal{B}$. A more rigorous treatment of this technicality can be found in [15].

In order to simplify our notation, we also define the following quantities:

Definition 1.3: Let $\mathbf{a}$ and $\mathbf{b}$ be stochastic processes. The following is the definition of (mutual) information rate[5]:

$$I_\infty(\mathbf{a};\mathbf{b}) = \limsup_{N\to\infty} \frac{I(\mathbf{a}_0^{N-1};\mathbf{b}_0^{N-1})}{N}$$

Definition 1.4: For a given stochastic process, we also define entropy rate as:

$$h_\infty(\mathbf{a}) = \limsup_{N\to\infty} \frac{h(\mathbf{a}_m^{N-1})}{N-m} \quad (5)$$

where m is the time delay represented in Fig 2.
The use of information rates is motivated by its universality [3], i.e., it quantifies the rate at which information can be reliably transmitted through an arbitrary communication medium.

In this paper, we will refer to channels which are stochastic operators conforming to the following definition:

Definition 1.5: (**Channel**) Let $\mathcal{V}$ and $\mathcal{R}$ be given input and output alphabets, along with a stochastic process, denoted as $\mathbf{c}$, with alphabet $\mathcal{C}$. In addition, consider a causal map $f: \mathcal{C}^\infty \times \mathcal{V}^\infty \to \mathcal{R}^\infty$. The pair $(f,\mathbf{c})$ defines a channel. The following are examples of channels:

- **Additive white Gaussian channel:** $\mathcal{V} = \mathcal{R} = \mathcal{C} = \mathbb{R}$, $\mathbf{c}$ is an i.i.d. white Gaussian sequence with unit variance and $f(\mathbf{c},\mathbf{v})(k) = \mathbf{c}(k) + \mathbf{v}(k)$.
- **Binary symmetric channel, with error probability** p_e**:** $\mathcal{V} = \mathcal{R} = \mathcal{C} = \mathbb{Z}_2 = \{0,1\}$, $\mathbf{c}$ is an i.i.d sequence satisfying $\mathcal{P}(\mathbf{c}(k) = 1) = p_e$ and $f(\mathbf{c},\mathbf{v})(k) = \mathbf{c}(k) +_{mod2} \mathbf{v}(k)$

For any given channel $(f,\mathbf{c})$, the supremum of the achievable information rates is a fundamental quantity denoted as capacity [3]. The formal definition of capacity can be found in [3], for which the following holds:

$$\forall k, \sup_{p_{\mathbf{v}_0^k}} \frac{I\left(\mathbf{v}_0^k; \{f(\mathbf{v},\mathbf{c})\}_0^k\right)}{k} \leq C \quad (6)$$

where the supremum is taken over all allowable $\mathbf{v}_0^k$ and C represents the capacity of the channel $(f,\mathbf{c})$.

[5]Throughout the paper, for simplicity, we refer to mutual information rate simply as information rate.

E. Spectral Properties of Asymptotically Stationary Stochastic Processes

We adopt the following definition of asymptotic power spectral density.

Definition 1.6: A given zero mean real stochastic process $\mathbf{a}$ is asymptotically stationary if the following limit exists for every $\gamma \in \mathbb{N}$:

$$\bar{R}_a(\gamma) \overset{def}{=} \lim_{k\to\infty} \mathcal{E}[\mathbf{a}(k+\gamma)\mathbf{a}(k)] \quad (7)$$

We also use (7) to define the following asymptotic power spectral density:

$$\hat{F}_a(\omega) = \sum_{k=-\infty}^{\infty} \bar{R}_a(k) e^{-j\omega k} \quad (8)$$

II. Technical Framework and Assumptions

Regarding the general scheme of Fig 1, the following assumptions are made:

- $\mathbf{w}$ is a scalar ($w(k) \in \mathbb{R}$), unit variance, identically and independently distributed stochastic process. For each k, $\mathbf{w}(k)$ is distributed according to a density $p_{\mathbf{w}}$, satisfying $|h_\infty(\mathbf{w}(k))| < \infty$.
- G is an all-pole stable filter of the form:

$$G(z) = \frac{\alpha}{1 - \sum_{m=1}^{p} a_m z^{-m}} \quad (9)$$

  where $p \geq 1$, a_i and $\alpha > 0$ are given. We chose this form of G, as a way to model the disturbance $\mathbf{d}$, not only because it is convenient that G^{-1} is well defined and causal, but also because there exists a very large class of power spectral densities that can be arbitrarily well approximated by $|G(\omega)|^2$ [22]. In addition, we assume that G has zero initial conditions.
- given n, P is a single input plant with state $x(k) \in \mathbb{R}^n$, which satisfies the following state-space equation:

$$\mathbf{x}(k+1) = \begin{bmatrix} x_u(k+1) \\ x_s(k+1) \end{bmatrix} = \begin{bmatrix} A_u & 0 \\ 0 & A_s \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} b_u \\ b_s \end{bmatrix} \mathbf{e}(k) \quad (10)$$

$$\mathbf{y}(k) = C\mathbf{x}(k), |\lambda_i(A_u)| \geq 1, |\lambda_i(A_s)| < 1 \text{ and } k \geq 0$$

  The state partitions $\mathbf{x}_u$ and $\mathbf{x}_s$ represent the unstable and stable open-loop dynamics, respectively. In addition, the initial state $\mathbf{x}(0)$ is a random variable satisfying $|h(\mathbf{x}_u(0))| < \infty$.
- $\mathbf{q}$, $\mathbf{w}$, $\mathbf{x}(0)$ and $\mathbf{c}$ are independent, where $\mathbf{c}$ represents the channel noise according to definition 1.5.
- the measurement noise $\mathbf{q}$ is such that the following holds:

$$I(\mathbf{x}(0); \mathbf{u}_0^{m-1}) < \infty$$

  meaning that the controller does not have access to an *exact description* of the initial state of the plant. Since $\mathbf{u}_0^{m-1} = \mathbf{e}_0^{m-1}$, we could have equivalently required that $I(\mathbf{x}(0); \mathbf{e}_0^{m-1}) < \infty$.
- $\mathbf{e}$ is a scalar ($s(k) \in \mathbb{R}$) stochastic process for which $\mathbf{e}_{k_{min}}^{k_{max}}$ has a probability density function, for every finite $k_{min}, k_{max} \geq m$.

A. Performance measures using entropy rates and asymptotic power spectra

In section III, which comprises our most general results, we characterize limits of performance by means of a lower bound to the difference $h_\infty(\mathbf{e}) - h_\infty(\mathbf{d})$. In standard texts, such as [20], [3], the entropy rate, of a given stochastic process, is interpreted as a measure of randomness or power. We use entropy rates to gage performance, not only because it is technically convenient, by allowing us to derive inequalities involving the information rate at the remote preview system (RPS) block, but also because it is a fundamental quantity which can be related to other, more common, measures of performance.

1) Definition of a sensitivity-like function:

Definition 2.1: If the stochastic process $\mathbf{e}$, represented in Fig 1, is asymptotic stationary then we define the following sensitivity-like function:

$$S(\omega) \stackrel{def}{=} \limsup_{\sigma_q \to 0} \sqrt{\frac{\hat{F}_e(\omega)}{\hat{F}_d(\omega)}} \tag{11}$$

III. Fundamental limitations for the general case

In this section, we derive performance bounds for the scheme[6] of Fig 2. In such case, the remote preview system is constructed by means of an arbitrary channel and a general encoder.

In the rest of the paper, we adopt the following assumptions:

- (A1) E and K are causal operators defined in the appropriate spaces, i.e., the output of E must belong to the channel input alphabet, which might be discrete or continuous. Similarly, the output of the channel must be defined in the r-input alphabet of K (see Fig 2).
- (A2) (**Feedback well-posedness**) we assume that the feedback system is well-posed, i.e., that there exists a causal operator J such that the following is well defined:

$$\mathbf{u}(k) = J(\mathbf{x}(0), \mathbf{r}, \mathbf{d}, \mathbf{q})(k), k \geq 0 \tag{12}$$

Notice that, at each time step k, we can also define a time varying J_k such that:

$$J_k(\mathbf{x}(0), \mathbf{r}_0^k, \mathbf{d}_0^{k-1}, \mathbf{q}_0^k) \stackrel{def}{=} J(\mathbf{x}(0), \mathbf{r}, \mathbf{d}, \mathbf{q})(k), k \geq 0$$

A. Derivation of a general bound involving entropy rates

As we have discussed in section II-A, we use $h_\infty(\mathbf{e}) - h_\infty(\mathbf{d})$ as a performance measure for the most general case, where we do not require $\mathbf{e}$ to be asymptotic stationary. The following Theorem provides an universal lower bound for $h_\infty(\mathbf{e}) - h_\infty(\mathbf{d})$ as a function of the unstable poles of P and the capacity of the remote preview channel. All of the remaining results in this section are, in one way or another, consequences of such universal lower bound.

[6]Although our results are valid for the general scheme of Fig 1, for simplicity, we consider the concrete scenario depicted in Fig 2.

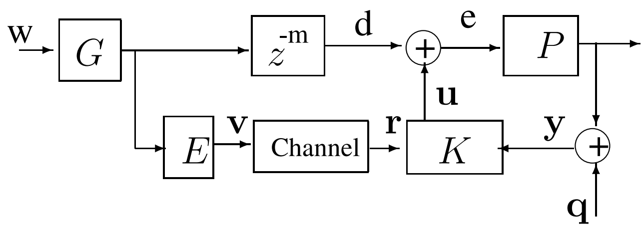

Fig. 2. Structure of a remote preview system, using a general communication channel.

Theorem 3.1: Consider the feedback interconnection represented in Fig 2. In addition, assume that the plant (10) satisfies $\sup_k \mathcal{E}[(\mathbf{x}(k))^T \mathbf{x}(k)] < \infty$. For any encoder E and controller K, satisfying (A1) and (A2), the following is true:

$$h_\infty(\mathbf{e}) - h_\infty(\mathbf{d}) \geq \sum_{i=1}^{n} \max\{0, \log(|\lambda_i(A)|)\} - C \tag{13}$$

where C represents the capacity [3] of the remote preview channel.

The proof of Theorem 3.1 can be found in section IV-B.2. In addition, section IV comprises not only all the details of the proof, but it also contains preliminary results, which clarify aspects such as the role of causality and stability.

B. Expressing performance limitations by means of asymptotic power spectral densities: an extension to Bode's integral formula

Under asymptotic stationary assumptions, in Theorem 3.4 we ascribe a frequency domain interpretation to Theorem 3.1.

We start with the following lemma, which establishes a connection between $h_\infty(\mathbf{e})$ and its asymptotic power spectral density $\hat{F}_e$.

Lemma 3.2: If $\mathbf{e}$ is an asymptotically stationary process[7] then the following holds:

$$\frac{1}{4\pi} \int_{-\pi}^{\pi} \log(2\pi e \hat{F}_e(\omega)) d\omega \geq h_\infty(\mathbf{e}) \tag{14}$$

Proof: Let $\tilde{\mathbf{e}}$ be a zero-mean Gaussian stochastic process such that $\mathcal{E}[\tilde{\mathbf{e}}(k+\gamma)\tilde{\mathbf{e}}(k)] = \mathcal{E}[\mathbf{e}(k+\gamma)\mathbf{e}(k)]$ holds. In terms of conditional differential entropy, we can write:

$$h_\infty(\mathbf{e}) \leq \limsup_{N \to \infty} \frac{h(\tilde{\mathbf{e}}_m^{N-1})}{N-m} \tag{15}$$

where we used the fact that Gaussian distributions maximize differential entropy [3], for a given covariance matrix. In addition, we can also write [3]:

$$\forall \gamma \in \mathbb{N}_+, \limsup_{N \to \infty} \frac{h(\tilde{\mathbf{e}}_m^{N-1})}{N-m} \leq \limsup_{N \to \infty} \frac{\sum_{k=m+\gamma}^{N-1} h(\tilde{\mathbf{e}}(k)|\tilde{\mathbf{e}}_{k-\gamma}^{k-1})}{N-m} \tag{16}$$

Now choose $\bar{\mathbf{e}}$ as a zero mean Gaussian stationary stochastic process with an auto-correlation given by $\bar{R}_e(\gamma)$, the

[7]In more rigorous terms, we should also require that $\hat{F}_e$ is Lebesgue integrable. More details can be found in [10], pp. 64-65.

asymptotic auto-correlation of $\mathbf{e}$. We can use such limit auto-correlation and substitute (16) in (15) to obtain:

$$\forall \gamma \in \mathbb{N}_+, h_\infty(\mathbf{e}) \leq h(\bar{\mathbf{e}}(\gamma)|\bar{\mathbf{e}}_0^{\gamma-1}) \tag{17}$$

as well as the following limit:

$$h_\infty(\mathbf{e}) \leq \lim_{\gamma\to\infty} h(\bar{\mathbf{e}}(\gamma)|\bar{\mathbf{e}}_0^{\gamma-1}) \tag{18}$$

On the other hand, we know that [3]:

$$\lim_{\gamma\to\infty} h(\bar{\mathbf{e}}(\gamma)|\bar{\mathbf{e}}_0^{\gamma-1}) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \log(2\pi e \hat{F}_e(\omega))d\omega \tag{19}$$

□

By means of lemma 3.2 and Theorem 3.1, we arrive at the following lemma:

Lemma 3.3: Consider the feedback interconnection represented in Fig 2. In addition, assume that the plant (10) satisfies $\sup_k \mathcal{E}[\mathbf{x}^T(k)\mathbf{x}(k)] < \infty$. For any encoder E and controller K, satisfying (A1)-(A2) and such that $\mathbf{e}$ is asymptotic stationary, the following is true:

$$\frac{1}{4\pi}\int_{-\pi}^{\pi} \log(2\pi e \hat{F}_e(\omega))d\omega \geq \sum_{i=1}^{n} \max\{0, \log(|\lambda_i(A)|)\} - C + h_\infty(\mathbf{d}) \tag{20}$$

where C represents the capacity [3] of the RPS channel.

We can use lemma 3.3 to state the following Theorem, which provides a fundamental limitation in terms of S.

Theorem 3.4: Consider the feedback interconnection represented in Fig 2. In addition, assume that the plant (10) satisfies $\sup_k \mathcal{E}[\mathbf{x}^T(k)\mathbf{x}(k)] < \infty$. If the encoder E and the controller K are such that (A1)-(A2) are satisfied and $\mathbf{e}$ is asymptotic stationary then the following is true:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(\sqrt{2\pi e}S(\omega)\right)d\omega \geq \sum_{i=1}^{n} \max\{0, \log(|\lambda_i(A)|)\} - C + h_\infty(\mathbf{w}) \tag{21}$$

where C represents the capacity [3] of the RPS channel. In addition, if $\mathbf{w}$ is Gaussian then (21) is given by:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(S(\omega)\right)d\omega \geq \sum_{i=1}^{n} \max\{0, \log(|\lambda_i(A)|)\} - C \tag{22}$$

Proof: Since $\mathbf{w}$ is i.i.d., we can write the following equality:

$$h_\infty(\mathbf{d}) = \lim_{k\to\infty} h(\mathbf{d}(k)|\mathbf{d}_m^{k-1}) = log(\alpha) + h_\infty(\mathbf{w}) \tag{23}$$

where we used the facts that G has zero initial conditions and that, in the time-domain, G is represented as:

$$\mathbf{d}(k) = \alpha\mathbf{w}(k) + \sum_{i=1}^{p} a_i \mathbf{d}(k-i)$$

Using the fact that $\int_{-\pi}^{\pi}\log|G(e^{j\omega})|d\omega = 2\pi\alpha$, we know that (23) can be re-written as:

$$h_\infty(\mathbf{d}) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \log(|G(e^{j\omega})|^2)d\omega + h_\infty(\mathbf{w}) \tag{24}$$

Since $\mathbf{d}$ is asymptotically stationary and $\mathbf{w}$ is unit variance, we also have that $\hat{F}_d(\omega) = |G(e^{j\omega})|^2$ and (24) can be written as:

$$h_\infty(\mathbf{d}) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \log(\hat{F}_d(\omega))d\omega + h_\infty(\mathbf{w}) \tag{25}$$

From Lemma 3.3 and (25), we arrive at:

$$\forall \sigma_q > 0, \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(\sqrt{2\pi e\frac{\hat{F}_e(\omega)}{\hat{F}_d(\omega)}}\right)d\omega \geq \sum_{i=1}^{n} \max\{0, \log(|\lambda_i(A)|)\} - C + h_\infty(\mathbf{w}) \tag{26}$$

After taking limits, and from the definition of $S(\omega)$, we get:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log[\sqrt{2\pi e}S(\omega)]d\omega \geq \limsup_{\sigma_q\to 0} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(\sqrt{2\pi e\frac{\hat{F}_e(\omega)}{\hat{F}_d(\omega)}}\right)d\omega \tag{27}$$

The proof of (21) follows from (27) and (26). If $\mathbf{w}$ is Gaussian then $h_\infty(\mathbf{w}) = \frac{1}{2}\log(2\pi e)$ and (22) follows from (21) by direct substitution. □

IV. Proof of Theorem 3.1

In this section, we provide not only technical preliminaries in Information Theory, but we also derive the auxiliary results which lead to the proof of Theorem 3.1, given in subsection IV-B.2.

A. Preliminaries on Information Theory

Using Theorem 2.1.2 of [21], we know that if $\log p_a$ and $\log p_{a|b}$ are integrable with respect to the probability measure induced in $\mathcal{A}\times\mathcal{B}$ then we can compute $I(\mathbf{a};\mathbf{b})$ as:

$$I(\mathbf{a};\mathbf{b}) = h(\mathbf{a}) - h(\mathbf{a}|\mathbf{b}) \tag{28}$$

In this paper, if we use (28) then, implicitly, we assume that $\log p_a$ and $\log p_{a|b}$ are integrable with respect to the probability measure induced in $\mathcal{A}\times\mathcal{B}$.

The following is a list of properties used in the subsequent sections. The proof of such properties may be found in [21] and, in some cases, in [3]: **(P1):** $I(\mathbf{a};\mathbf{b}) = I(\mathbf{b};\mathbf{a}) \geq 0$ and $I(\mathbf{a};\mathbf{b}|\mathbf{c}) \geq 0$; **(P2) Kolmogorov's formula** [8] **(equation 3.6.6 in [21]):**

$$I((\mathbf{a},\mathbf{b});\mathbf{c}|\mathbf{d}) = I(\mathbf{b};\mathbf{c}|\mathbf{d}) + I(\mathbf{a};\mathbf{c}|(\mathbf{b},\mathbf{d}))$$

(P3): Theorem 3.7.1 in [21]: If f and g are measurable functions in the appropriate probability spaces then $I(f(\mathbf{a});g(\mathbf{b})|\mathbf{c}) \leq I(\mathbf{a};\mathbf{b}|\mathbf{c})$ and equality holds if f and g are invertible[9]; **(P4):** From property (P3), we conclude that $I(\mathbf{a};(\mathbf{b},\mathbf{c})|\mathbf{d}) = I(\mathbf{a};(\mathbf{b}-\mathbf{c},\mathbf{c})|\mathbf{d})$. Using (P2), such

[8] Notice that equation 3.6.3 in [21] has a typographic mistake. On the left hand side of the equality, the correct is $I(\xi,\zeta)$

[9] In [21] equality is guaranteed for everywhere dense f and g. Every time we say that a function is invertible in this context we are implicitly assuming that it is everywhere dense.

equality also leads to $I(\mathbf{a};\mathbf{b}|(\mathbf{c},\mathbf{d})) = I(\mathbf{a};\mathbf{b}-\mathbf{c}|(\mathbf{c},\mathbf{d}))$; **(P5):** By means of (P1) and (28), we infer that $h(\mathbf{a}) \geq h(\mathbf{a}|\mathbf{b})$, where equality holds if and only if $\mathbf{a}$ and $\mathbf{b}$ are independent. Likewise, we can use properties (P1)-(P2) to state that $I(\mathbf{a};(\mathbf{b},\mathbf{c})) \geq I(\mathbf{a};\mathbf{b})$, which can be used with (28) to derive $h(\mathbf{a}|\mathbf{b}) \geq h(\mathbf{a}|(\mathbf{b},\mathbf{c}))$; **(P6):** Using a change of variables in the integrals of definition 1.2, we reckon that if $f:\mathcal{B}\to\mathcal{A}$ is any given function then $h(\mathbf{a}|\mathbf{b}) = h(\mathbf{a}-f(\mathbf{b})|\mathbf{b})$; **(P7) [3]:** If $\mathbf{a}$ has a finite covariance matrix Σ_a then $h(\mathbf{a}) \leq \frac{1}{2}\log((2\pi e)^n \det(\Sigma_a))$.

B. Auxiliary Results and the Proof of Theorem 3.1

The central result is presented in the following lemma, stating that $h_\infty(\mathbf{e})$ is lower bounded by $h_\infty(\mathbf{d}) + \liminf_{N\to\infty} \frac{I(\mathbf{x}(0);\mathbf{e}_0^{N-1})}{N} - I_\infty(\mathbf{r};\mathbf{d})$, where $I_\infty(\mathbf{r};\mathbf{d})$ quantifies the information rate flowing through the remote preview channel.

Lemma 4.1: **(Main entropy rate inequality)** Consider the feedback interconnection represented in Fig 2. For any encoder E and controller K, satisfying the assumptions (A1) and (A2), the following holds:

$$h_\infty(\mathbf{e}) - h_\infty(\mathbf{d}) \geq \liminf_{k\to\infty} \frac{I(\mathbf{x}(0);\mathbf{e}_0^k)}{k} - I_\infty(\mathbf{r};\mathbf{d}) \quad (29)$$

Proof: We start by choosing arbitrary $k \geq m$ and using the fact that G has zero initial conditions to write:

$$h_\infty(\mathbf{d}) = h(\mathbf{d}(k)|\mathbf{d}_0^{k-1}) \quad (30)$$

By means of (28), (P2) and (30), we obtain:

$$h_\infty(\mathbf{d}) = h(\mathbf{d}(k)|(\mathbf{d}_0^{k-1},\mathbf{x}(0),\mathbf{u}_0^k,\mathbf{q}_0^k)) + I((\mathbf{x}(0),\mathbf{u}_0^k,\mathbf{q}_0^k);\mathbf{d}(k)|\mathbf{d}_0^{k-1}) \quad (31)$$

We proceed by re-writing each of the terms in the right hand side of (31). By means of (P6) and using the fact that $e(k) = d(k) + u(k)$, we find that:

$$\begin{aligned} &h(\mathbf{d}(k)|(\mathbf{d}_0^{k-1},\mathbf{x}(0),\mathbf{u}_0^k,\mathbf{q}_0^k)) = \\ &h(\mathbf{e}(k)|(\mathbf{d}_0^{k-1},\mathbf{x}(0),\mathbf{e}_0^{k-1},\mathbf{u}(k),\mathbf{q}_0^k)) \underset{(P5)}{\leq} \\ &h(\mathbf{e}(k)|(\mathbf{e}_0^{k-1},\mathbf{x}(0))) \underset{(28)\text{ and }(P2)}{=} \\ &h(\mathbf{e}(k)|\mathbf{e}_0^{k-1}) - I(\mathbf{e}(k);\mathbf{x}(0)|\mathbf{e}_0^{k-1}) \end{aligned} \quad (32)$$

Using the well-posedness assumption (A2), together with (P2)-(P3), we can bound the second term on the right hand side of (31) by means of the following inequality:

$$I((\mathbf{x}(0),\mathbf{u}_0^k,\mathbf{q}_0^k);\mathbf{d}(k)|\mathbf{d}_0^{k-1}) \leq I((\mathbf{x}(0),\mathbf{r}_0^k,\mathbf{q}_0^k);\mathbf{d}(k)|\mathbf{d}_0^{k-1}) \quad (33)$$

which, since $\mathbf{x}(0)$, $\mathbf{q}_0^k$ and $(\mathbf{r}_0^k,\mathbf{d}_0^k)$ are mutually independent, can be expressed as:

$$I((\mathbf{x}(0),\mathbf{u}_0^k,\mathbf{q}_0^k);\mathbf{d}(k)|\mathbf{d}_0^{k-1}) \leq I(\mathbf{r}_0^k;\mathbf{d}(k)|\mathbf{d}_0^{k-1}) \quad (34)$$

By direct substitution of (34) and (32) into (31), we arrive at:

$$h_\infty(\mathbf{d}) \leq h(\mathbf{e}(k)|\mathbf{e}_0^{k-1}) - I(\mathbf{e}(k);\mathbf{x}(0)|\mathbf{e}_0^{k-1}) + I(\mathbf{r}_0^k;\mathbf{d}(k)|\mathbf{d}_0^{k-1}) \quad (35)$$

Now, choose arbitrary $N > m$ so that we have the following inequality based on (35):

$$h_\infty(\mathbf{d}) \leq \frac{1}{N-m}\sum_{k=m}^{N-1} I(\mathbf{r}_0^k;\mathbf{d}(k)|\mathbf{d}_0^{k-1}) + \frac{1}{N-m}\left(\sum_{k=m}^{N-1} h(\mathbf{e}(k)|\mathbf{e}_0^{k-1}) - \sum_{k=m}^{N-1} I(\mathbf{e}(k);\mathbf{x}(0)|\mathbf{e}_0^{k-1})\right) \quad (36)$$

From (P3) we know that for any $k \leq N-1$ the following holds:

$$I(\mathbf{r}_0^k;\mathbf{d}(k)|\mathbf{d}_0^{k-1}) \leq I(\mathbf{r}_0^{N-1};\mathbf{d}(k)|\mathbf{d}_0^{k-1})$$

which, together with (P2), (P5) and (36), leads to:

$$h_\infty(\mathbf{d}) \leq \frac{1}{N-m}\left(h(\mathbf{e}_m^{N-1}) - I(\mathbf{e}_0^{N-1};\mathbf{x}(0))\right) + \frac{1}{N-m}\left(I(\mathbf{r}_0^{N-1};\mathbf{d}_0^{N-1}) + I(\mathbf{e}_0^{m-1};\mathbf{x}(0))\right) \quad (37)$$

Using the fact that $I(\mathbf{e}_0^{m-1};\mathbf{x}(0))$ is finite, we conclude the proof by considering the limit as $N\to\infty$ in (37). □

1) Incorporating Stability: According to the following lemma, stability suffices to guarantee that $\liminf_{N\to\infty}\frac{1}{N}I(\mathbf{e}_0^{N-1};\mathbf{x}(0)) \geq \sum_{i=1}^n \max\{0,\log(|\lambda_i(A)|)\}$, where A is the dynamic matrix of P, as described by (10). Such result follows from [25], [27], [18] and is the last step towards proving Theorem 3.1.

Lemma 4.2: Let $\mathbf{x}(k)$ be the solution of the state-space equation (10). If the system satisfies $\sup_k \mathcal{E}[(\mathbf{x}(k))^T\mathbf{x}(k)] < \infty$ then the following holds:

$$\liminf_{N\to\infty}\frac{I(\mathbf{e}_0^{N-1};\mathbf{x}(0))}{N} \geq \sum_{i=1}^n \max\{0,\log(|\lambda_i(A)|)\} \quad (38)$$

Proof: If $A = A_s$ then we just use $I(\mathbf{e}_0^{N-1};\mathbf{x}(0)) \geq 0$. If $A \neq A_s$ then we consider the following homogeneous system:

$$\mathbf{x}_e(k+1) = A_u\mathbf{x}_e(k) + b_u\mathbf{e}(k),\ x_e(0) = 0 \quad (39)$$

and define the estimate $\hat{\mathbf{x}}(k) = A_u^{-k}\mathbf{x}_e(k)$. Since $\mathbf{x}_u(k) = \mathbf{x}_e(k) + A_u^k\mathbf{x}_u(0) = A_u^k(\hat{\mathbf{x}}(k) - \mathbf{x}_u(0))$, we know that:

$$k\log(|\det(A_uA_u^T)|) + \log(\det(R_{\mathbf{x}_{error}}(k))) = \log(\det(R_{\mathbf{x}_u}(k,k))) < \beta < \infty \quad (40)$$

where $\mathbf{x}_{error}(k) = \hat{\mathbf{x}}(k) - \mathbf{x}_u(0)$. Since $\hat{x}(k)$ is a function of s_0^k, we have that:

$$I(\mathbf{x}(0);\mathbf{e}_0^{N-1}) \geq I(\mathbf{x}_u(0);\mathbf{e}_0^{N-1}) \geq h(\mathbf{x}_u(0)) - h(\hat{\mathbf{x}}(N-1) - \mathbf{x}_u(0)) \quad (41)$$

But, from (P7) we know that:

$$\limsup_{N\to\infty}\frac{h(\hat{\mathbf{x}}(N-1)-\mathbf{x}_u(0))}{N} \leq \limsup_{N\to\infty}\frac{\log(\det(R_{\mathbf{x}_{error}}(N-1)))}{2N} \quad (42)$$

As a consequence, we can use (40) to get $\limsup_{N\to\infty} \frac{h(\hat{\mathbf{x}}(N-1)-\mathbf{x}_u(0))}{N} \leq -\log(|\det(A_u)|)$. The proof follows by direct substitution $\Box$.

2) Proof of Theorem 3.1: The following remark constitutes a proof of Theorem 3.1, where the statement is repeated for convenience.

Remark 4.1: **(Proof of Theorem 3.1)** Consider the feedback interconnection represented in Fig 2. In addition, assume that the plant (10) is stabilized, i.e., $\sup_k \mathcal{E}[(\mathbf{x}(k))^T\mathbf{x}(k)] < \infty$ holds. For any encoder E and controller K, satisfying (A1) and (A2), the following is true:

$$h_\infty(\mathbf{e}) - h_\infty(\mathbf{d}) \geq \sum_{i=1}^{n} \max\{0, \log(|\lambda_i(A)|)\} - C \quad (43)$$

where C represents the capacity [3] of the remote preview channel.

Proof: By direct substitution of (38) into (29), we obtain:

$$h_\infty(\mathbf{e}) - h_\infty(\mathbf{d}) \geq \sum_{i=1}^{n} \max\{0, \log(|\lambda_i(A)|)\} - I_\infty(\mathbf{r};\mathbf{d}) \quad (44)$$

The proof follows from the definition of channel capacity [3], i.e., $I_\infty(\mathbf{r};\mathbf{v}) \leq C$ and by the data processing inequality $I_\infty(\mathbf{r};\mathbf{d}) \leq I_\infty(\mathbf{r};\mathbf{v}) \leq C$.$\Box$

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