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$\mathcal{H}_\infty$ Control of Nonlinear Systems: A Convex Characterization

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***Abstract*—The nonlinear $\mathcal{H}_\infty$-control problem is considered with an emphasis on developing machinery with promising computational properties. The solutions to $\mathcal{H}_\infty$-control problems for a class of nonlinear systems are characterized in terms of nonlinear matrix inequalities which result in convex problems. The computational implications for the characterizations are discussed.**

I. INTRODUCTION

The nonlinear $\mathcal{H}_\infty$-control problem has been considered extensively in a state-space framework [20], [10], [1], [14]. Basically, in those treatments, the (dynamic) output feedback $\mathcal{H}_\infty$-controllers have separation structures, and necessary and sufficient conditions for the $\mathcal{H}_\infty$-control problem to have solutions are characterized in terms of Hamilton–Jacobi equations or inequalities [20], [10], [1], [14], [21]. Specifically, a local output feedback $\mathcal{H}_\infty$-controller and a class of parameterized local $\mathcal{H}_\infty$ controllers are designed based on the required local solutions of some Hamilton–Jacobi equations of inequalities [10], [14]; also the fact that there exist output feedback $\mathcal{H}_\infty$-controllers (with separation structures) implies the solvability of two Hamilton–Jacobi equations or inequalities [1], [21]. Some efforts have been made to characterize the global solutions; a one-inequality sufficient and necessary condition for global solutions is given by Helton and Zhan in [7]; the necessary conditions can be further refined to two Hamilton–Jacobi inequalities [1]. Whence, one of the major concerns in the state-space nonlinear $\mathcal{H}_\infty$-control theory is the computation issue involving in solving these Hamilton–Jacobi (partial differential) equations (HJE's) or inequalities (HJI's), progress along this line would be beneficial to applications of nonlinear $\mathcal{H}_\infty$-control theory. For example, Huang and Lin proposed a systematic procedure to find Taylor series approximations to the solutions of the HJE's [9] (see also [20]).

In this paper, we propose an alternative approach with promising computational properties to the nonlinear $\mathcal{H}_\infty$-control problem. This is motivated by the fact that, essentially, the linear $\mathcal{H}_\infty$-control problem can be characterized as a convex problem which has some appealing computational properties [18], [3] (see also [17], [19], [13], [6], [11] for the treatments in linear case in terms of linear matrix inequalities (LMI's), which result in convex problems). We therefore examine the convexity of the nonlinear $\mathcal{H}_\infty$-control problem and characterize the solutions in terms of nonlinear matrix inequalities (NLMI's) instead of the Hamilton–Jacobi equations or inequalities. Both state feedback and output feedback solutions are derived. In the output feedback case, the $\mathcal{H}_\infty$-controllers are not required to have separation structures; some necessary conditions are characterized in terms of three algebraic NLMI's. It is also confirmed that the three-NLMI characterization is sufficient for local solutions. It is noted that the algebraic NLMI's are in fact state-dependent LMI's,

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therefore, some convex optimization methods for solving LMI's can be possibly used in the practical computation for solving NLMI's. Unfortunately, unlike the linear case, the solution of the NLMI's by themselves are not sufficient to guarantee the existence of the required controller, some additional condition is required, and the computational implications of the required additional constraints on the NLMI solutions are not totally clear at this moment. This issue is discussed more in the body of the paper.

The remainder of this paper is organized as follows: In Section II, some background material related to the $\mathcal{L}_2$-gains analysis is provided and an NLMI characterization of $\mathcal{L}_2$-gains is given. In Section III, the $\mathcal{H}_\infty$-control problem is stated, and some assumptions on the system structures are made. In Section IV, the main results of this paper, i.e., solutions to the output feedback $\mathcal{H}_\infty$-control problem, are given; the solvability of this problem is characterized by three NLMI's, and it is further shown that under some (weak) separation structure assumptions for the $\mathcal{H}_\infty$-controllers, the solvability of the output-feedback $\mathcal{H}_\infty$-control problem implies the solvability by static state-feedback. In Section V, the computational implications for NLMI characterizations is examined. Some required technical material is reviewed in the appendix.

The following conventions are made in this paper. $\boldsymbol{R}$ is the set of real numbers, $\boldsymbol{R}^+ := [0, \infty) \subset \boldsymbol{R}$. $\boldsymbol{R}^n$ is n-dimensional real Euclidean space; $\|\cdot\|$ stands for the Euclidean norm. For $\mathcal{B}_r$, it is understood to be the open ball in some Euclidean space with some radius $r > 0$ which is measured by Euclidean norm. $\boldsymbol{X}$ (or $\boldsymbol{X}_o$) is the state set which is a convex open bounded subset of some Euclidean space and contains the origin. $\mathcal{L}_2(\boldsymbol{R}^+)$ stands for the space of measurable (vector-valued) functions $u: \boldsymbol{R}^+ \to \boldsymbol{R}^m$ such that $\int_{\boldsymbol{R}^+} \|u(t)\|^2 \, dt < \infty$. $\boldsymbol{R}^{n\times m}$ ($\boldsymbol{C}^{n\times m}$) is the set of all $n \times m$ real (complex) matrices. The transpose of some matrix $M \in \boldsymbol{R}^{n\times n}$ is denoted by M^T. By $P > 0$ ($P \geq 0$) for some Hermitian matrix $P \in \boldsymbol{R}^{n\times n}$ or ($\boldsymbol{C}^{n\times m}$) we mean that the matrix is (semi)positive definite. A function is said to be of class $\boldsymbol{C}^k$ if it is continuously differentiable k times; so $\boldsymbol{C}^0$ stands for the class of continuous functions.

II. NLMI Characterizations of $\mathcal{H}_\infty$-Performances

In this section, some background material about $\mathcal{L}_2$-gain analysis of nonlinear systems is provided. The $\mathcal{L}_2$-gains of a nonlinear system are characterized in terms of NLMI's. The reader is referred to Willems [22], van der Schaft [20], and Lu and Doyle [14] for more characterizations.

Consider the following input-affine nonlinear time-invariant (NLTI) system

$$G: \begin{cases} \dot{x} = A(x)x + B(x)w \\ z = C(x)x + D(x)w \end{cases} \tag{1}$$

where $x \in \boldsymbol{R}^n$ is state vector and $w \in \boldsymbol{R}^p$ and $z \in \boldsymbol{R}^q$ are input and output vectors, respectively. We will assume A, B, C, D are $\boldsymbol{C}^0$ matrix-valued functions of suitable dimensions. From now on we will assume that the system evolves on a convex open bounded subset $\boldsymbol{X} \subset \boldsymbol{R}^n$ containing the origin. Thus, $0 \in \boldsymbol{R}^n$ is an equilibrium of the system with $w = 0$. The performance of system (1) is measured in terms of its $\mathcal{L}_2$-gain in this paper.

Definition 2.1: System (1) with initial state $x(0) = 0$ is said to have $\mathcal{L}_2$-gain less than or equal to γ for some $\gamma > 0$ if

$$\int_0^T \|z(t)\|^2 \, dt \leq \gamma^2 \int_0^T \|w(t)\|^2 \, dt \tag{2}$$

for all $T \geq 0$ and $w(t) \in \mathcal{L}_2[0, T]$, as long as the state $x(t) \in \boldsymbol{X}$ for $t \in [0, T]$.

The following theorem characterizes $\mathcal{L}_2$-gains for a class of nonlinear systems in terms of NLMI's.

Theorem 2.2: Consider system G given by (1), suppose $I - D^T(x)D(x) > 0$. Given any $\boldsymbol{C}^0$ matrix-valued function $P: \boldsymbol{X} \to \boldsymbol{R}^{n\times n}$, the following two inequalities are equivalent:

1) P satisfies (3) found at the bottom of the page for all $x \in \boldsymbol{X}$.
2) P satisfies

$$\hat{\mathcal{M}}(P, x) := \begin{bmatrix} A^T(x)P(x) + P^T(x)A(x) & P^T(x)B(x) & C^T(x) \\ B^T(x)P(x) & -I & D^T(x) \\ C(x) & D(x) & -I \end{bmatrix} < 0 \tag{4}$$

for all $x \in \boldsymbol{X}$.

In addition, if there are a positive definite $\boldsymbol{C}^0$ matrix-valued function $P: \boldsymbol{X} \to \boldsymbol{R}^{n\times n}$ satisfying any of the above inequalities and a function $V: \boldsymbol{X} \to \boldsymbol{R}$ such that $\partial V/\partial x(x) = 2xP(x)$, then the system has $\mathcal{L}_2$-gain ≤ 1 and is asymptotically stable.

Proof: The standard result of Schur complements yields $\mathcal{M}(P, x) < 0$ if and only if $\hat{\mathcal{M}}(P, x) < 0$, since $I - D^T(x)D(x) > 0$. As for the later statement, again by Schur complement argument, we have that (3) implies the following Hamilton–Jacobi inequality

$$\begin{aligned} \mathcal{H}\left(\frac{\partial V}{\partial x}, x\right) &:= \frac{\partial V}{\partial x}(x)A(x)x + x^T C^T(x)C(x)x \\ &\quad + \left(\frac{1}{2}\frac{\partial V}{\partial x}(x)B(x) + x^T C^T(x)D(x)\right) \\ &\quad \cdot (I - D^T(x)D(x))^{-1} \\ &\quad \cdot \left(\frac{1}{2}B^T(x)\frac{\partial V^T}{\partial x}(x) + D^T(x)C(x)x\right) < 0 \end{aligned} \tag{5}$$

for all $x \in \boldsymbol{X}\backslash\{0\}$. The standard technique of completion of square then can be used to show that the system is asymptotically stable and has $\mathcal{L}_2$-gain ≤ 1 (see, for example, [20]). □

Remark 2.3: It is remarked that the above characterization of the $\mathcal{L}_2$-gain in terms of inequality (3) or (4) exhibits some appealing computational properties. It is noted that the left-hand side of inequality (3) or (4) is affine in $P(x)$, and all positive definite solutions form convex sets, i.e., the characterization is a convex condition. This trivial fact has only been exploited systematically in the linear case, but we hope that numerical techniques may be developed to exploit it in the nonlinear case as well. Inequalities (3) and (4) are actually state-dependent linear (or affine) matrix inequalities, but we will refer to them as NLMI's to emphasize their use in nonlinear problems.

Remark 2.4: It should be emphasized that the existence of a $\boldsymbol{C}^0$ matrix-valued function $P: \boldsymbol{X} \to \boldsymbol{R}^{n\times n}$ which satisfies any of the above NLMI's is not enough to guarantee the system to have $\mathcal{L}_2$-gain ≤ 1; it is additionally required that there exists a function $V: \boldsymbol{X} \to \boldsymbol{R}$ such that $\partial V/\partial x(x) = 2x^T P(x)$ (see Proposition 5.2 for a characterization of a class of matrix-valued function $P: \boldsymbol{X} \to \boldsymbol{R}^{n\times n}$ which satisfies this additional requirement).

$$\mathcal{M}(P, x) := \begin{bmatrix} A^T(x)P(x) + P^T(x)A(x) + C^T(x)C(x) & P^T(x)B(x) + C^T(x)D(x) \\ B^T(x)P(x) + D^T(x)C(x) & D^T(x)D(x) - I \end{bmatrix} < 0 \tag{3}$$

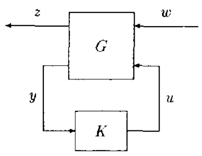

Fig. 1.

Remark 2.5: If there is a C^0 matrix-valued matrix P_0 such that $\mathcal{M}(P_0, x) < 0$ for $x \in X$, then by continuity of $\mathcal{M}$ with respect to x, there is another C^0 matrix-valued matrix P such that $\mathcal{M}(P, x) < 0$ and $\partial V/\partial x(x) = 2x^T P^T(x)$ for some C^1 function $V: \mathcal{B}_d \to R^+$ for some $d > 0$. In fact, a natural choice is a constant matrix $P = P_0(0)$ and $V(x) = x^T Px$. The same observation for Hamilton–Jacobi characterizations is made in [20].

It is noted that by Proposition 5.2, the C^1 function $V: X \to R$ which satisfies $\partial V/\partial x(x) = 2x^T P(x)$ for some positive definite matrix-valued function $P^T(x) = P(x) > 0$ and $V(0) = 0$ is positive definite on X. Now we conclude the above discussions by defining a stronger $\mathcal{H}_\infty$-performance.

Definition 2.6: The concerned system (1) is said to have strong $\mathcal{H}_\infty$-performance if there is a C^0 positive definite matrix-valued function $P(x) = P^T(x) > 0$ which satisfies any of inequalities (3) and (4) for all $x \in X$ such that $\partial V/\partial x(x) = 2x^T P(x)$ for some C^1 function $V: X \to R$.

Therefore, if system has a strong $\mathcal{H}_\infty$-performance, by Proposition 5.2 and Theorem 2.2, it has $\mathcal{L}_2$-gain ≤ 1. The conservativeness of the strong $\mathcal{H}_\infty$-performance characterized by the NLMI (3) or (3) is examined in [16].

III. $\mathcal{H}_\infty$-Control Problems

The feedback configuration for the $\mathcal{H}_\infty$-control synthesis problem is depicted in Fig. 1; where G is the nonlinear plant with two sets of inputs: the exogenous disturbance inputs w and the control inputs u, and two sets of outputs: the measured outputs y and the regulated outputs z. K is the controller to be designed. It is required that the feedback configuration be well posed. Both G and K are nonlinear time-invariant and can be realized as input-affine state-space equations

$$G: \begin{cases} \dot{x} = A(x)x + B_1(x)w + B_2(x)u \\ z = C_1(x)x + D_{11}(x)w + D_{12}(x)u \\ y = C_2(x)x + D_{21}(x)w + D_{22}(x)u \end{cases} \tag{6}$$

where $A, B_i, C_i, D_{ij} \in C^0$ $(i, j = 1, 2)$: x, w, u, z, and y are assumed to have dimensions n, p_1, p_2, q_1, and q_2, respectively (without loss of generality, it is assumed that $n + p_1 \geq q_2$ and $n + q_1 \geq p_2$

$$K: \begin{cases} \dot{\xi} + \hat{A}(\xi)\xi + \hat{B}(\xi)y \\ u = \hat{C}(\xi)\xi + \hat{D}(\xi)y \end{cases} \tag{7}$$

with $\hat{A}, \hat{B}, \hat{C}, \hat{D} \in C^0$. It is assumed that the feedback system (6)–(7) evolves on $X \times X_o$, where X and X_o are convex open bounded sets and contain the origins. The initial states for both plant and controller are $x(0) = 0$ and $\xi(0) = 0$.

In this paper, we shall consider the following version of $\mathcal{H}_\infty$-control problem.

(Strong) $\mathcal{H}_\infty$-Control Problem: Find a feedback controller K, if any, such that the closed-loop system has strong $\mathcal{H}_\infty$-performance and is asymptotically stable with $w = 0$. In this case, the feedback system has $\mathcal{L}_2$-gain ≤ 1, i.e.,

$$\int_0^T (\|w(t)\|^2 - \|z(t)\|^2)\, dt \geq 0$$

with $x(0) = 0$, $\xi(0) = 0$, for all $T \in R^+$ and $w \in \mathcal{L}_2(R^+)$, as long as the states $(x(t), \xi(t)) \in X \times X_o$ for $t \in [0, T]$. The strong $\mathcal{H}_\infty$-control problem is said to accept local solutions if the above requirements for the closed-loop system hold for $(x(t), \xi(t)) \in \mathcal{B}_r \times \mathcal{B}_s$ with some $r, s > 0$ for $t \in [0, T]$. The controllers to be sought in solving the above $\mathcal{H}_\infty$-control problem are called strong $\mathcal{H}_\infty$-controllers.

The following assumptions are made for the $\mathcal{H}_\infty$-control problem.

Assumption 3.1: Consider the given system (6) and controller (7):

1) $\text{RANK}\begin{bmatrix} B_2(x) \\ D_{12}(x) \end{bmatrix} = p_2$ for all $x \in X$.
2) $\text{RANK}[C_2(x) \quad D_{21}(x)] = q_2$ for all $x \in X$.
3) $D_{11}(x)D_{11}^T(x) < I$ for all $x \in X$.
4) $I - \hat{D}(\xi)D_{22}(x)$ is invertible for all $(x, \xi) \in X \times X_o$.

The first three regularity assumptions are for technical reason. The last assumption assures the well posedness of the feedback structure.

In the next few sections, we will characterize the solvability of the strong $\mathcal{H}_\infty$-control problem. Basically the treatment is divided into the following steps:

- Given a controller (7) for system (6) which yields a stable closed-loop system with strong $\mathcal{H}_\infty$-performance, characterize this closed-loop property in terms of NLMI (3) or (4) by Theorem 2.2. This NLMI depends on the coefficient matrix-valued functions of the controller.
- Further characterize the above NLMI such that the new characterizations are independent of the coefficient (matrix-valued) functions of the controller by Finsler's Theorem. The new characterization are three NLMI's.
- Examine the conditions under which the three NLMI's derived in the last step have the solutions that yield strong $\mathcal{H}_\infty$-control solution.

In the next section, the first two steps are mainly covered. The last step is treated in Section V.

IV. Solutions to $\mathcal{H}_\infty$-Control Problem

In this section, we will consider the general strong $\mathcal{H}_\infty$-control problem for the system given by (6) under assumptions A1), A2), and A3). The solvability conditions for the $\mathcal{H}_\infty$-control problem to have solutions are characterized in terms of NLMI's without assuming the controllers have separation structures.

Consider system (6) which evolves on X. Define

$$B(x) := [B_2^T(x) \quad D_{12}^T(x)], \qquad C(x) := [C_2(x) \quad D_{21}(x)].$$

Let $\mathcal{N}(B(x))$ be the distribution on X which annihilates the row vectors of $B(x)$. The main theorem of this section is stated as follows.

Theorem 4.1: Given system (6), suppose there is a solution to the output feedback (strong) $\mathcal{H}_\infty$ control problem. Then under Assumption 3.1 there are two C^0 positive definite matrix-valued functions $X, Y: X \to R^{n\times m}$ such that for all $x \in X \subset R^{n\times n}$:

i) (see (8) at the bottom of the next page) with $B_\perp: X \to R^{(n+q_1)\times(n+q_1-p_2)}$ such that $\mathcal{N}(B(x)) = \text{SPAN}\,(B_\perp(x))$.
ii) (see (9) at the bottom of the next page) with $C_\perp: X \to R^{(n+p_1)\times(n+p_1-q_2)}$ such that $\mathcal{N}(C(x)) = \text{SPAN}\,(C_\perp(x))$.
iii)

$$\begin{bmatrix} X(x) & I \\ I & Y(x) \end{bmatrix} \geq 0.$$

The proof of the main theorem is given next. The techniques used in the proof closely follows from [1], [21], [17], [19], [6], and [11].

Proof: Suppose there exists a strong $\mathcal{H}_\infty$-controller which is of input-affine form as follows

$$K: \begin{cases} \dot{\xi} = \hat{A}(\xi)\xi + \hat{B}(\xi)y \\ u = \hat{C}(\xi)\xi + \hat{D}(\xi)y \end{cases}$$

with $\hat{A}, \hat{B}, \hat{C}, \hat{D} \in C^0$. Suppose $\xi \in X_0 \subset R^{n_d}$ for some integer $n_d > 0$, where X_o is a convex open subset containing the origin. The closed-loop system evolves on $X \times X_o$. Now take $x_c = \begin{bmatrix} x \\ \xi \end{bmatrix}$ to be the state of the closed-loop system; define

$$R(x_c) := (I - \hat{D}(\xi)D_{22}(x))^{-1} \tag{10}$$

which is well defined for $(x, \xi) \in X \times X_o$ by assumption A4). The feedback system has the following description

$$\begin{cases} \dot{x}_c = A_c(x_c)x_c + B_c(x_c)w \\ z = C_c(x_c)x_c + D_c(x_c)w \end{cases}$$

with

$$A_c(x_c) = A^a(x) + B_2^a(x_c)F_c(x_c)C_2^a(x),$$

$$B_c(x_c) = B_1^a(x) + B_2^a(x_c)F_c(x_c)D_{21}^a(x), \tag{11}$$

$$C_c(x_c) = C_1^a + D_{12}^a(x)F_c(x_c)C_2^a(x),$$

$$D_c(x_c) = D_{11}^a(x) + D_{12}^a(x)F_c(x_c)D_{21}^a(x) \tag{12}$$

where

$$A^a(x) := \begin{bmatrix} A(x) & 0 \\ 0 & 0 \end{bmatrix}, \qquad B_1^a(x) := \begin{bmatrix} B_1(x) \\ 0 \end{bmatrix},$$

$$B_2^a(x_c) := \begin{bmatrix} B_2(x) & 0 \\ \hat{B}(\xi)D_{22}(x) & I \end{bmatrix}, \quad C_1^a(x) := [C_1(x) \quad 0]$$

$$D_{11}^a(x) := D_{11}(x), \quad D_{12}^a(x) := [D_{12}(x) \quad 0]$$

$$C_2^a(x) := \begin{bmatrix} C_2(x) & 0 \\ 0 & I \end{bmatrix}, \; D_{21}^a(x) := \begin{bmatrix} D_{21}(x) \\ 0 \end{bmatrix},$$

$$D_{22}^a(x) := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$F_c(x_c) := \begin{bmatrix} R(x_c)\hat{D}(\xi) & R(x_c)\hat{C}(\xi) \\ \hat{B}(\xi) & \hat{A}(\xi) \end{bmatrix}. \tag{13}$$

Since the feedback system has strong $\mathcal{H}_\infty$-performance, by Definition 2.6, there is a C^0 positive definite matrix-valued function $P_c(x_c)$ on $X \times X_0$ such that

$$\mathcal{M}_c(P_c, x_c) := \begin{bmatrix} A_c^T(x_c)P_c(x_c) + P_c(x_c)A_c(x_c) & P_c(x_c)B_c(x_c) & C_c^T(x_c) \\ B_c^T(x_c)P_c(x_c) & -I & D_c^T(x_c) \\ C_c(x_c) & D_c(x_c) & -I \end{bmatrix} < 0. \tag{14}$$

Re-organizing the left-hand side of the above NLMI yields

$$\mathcal{M}_c(P_c, x_c) = \mathcal{M}_a(P_c, x_c) + \tilde{C}^T(x_c)F_c^T(x_c)\tilde{B}(x_c)T_c(x_c) + T_c^T(x_c)\tilde{B}^T(x_c)F_c(x_c)\tilde{C}(x_c) < 0 \tag{15}$$

where (see (y) at the bottom of the page) and

$$T_c(x_c) = \begin{bmatrix} P_c(x_c) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

It follows from Lemma 7.3 that (15) holds only if the following two inequalities hold

$$\tilde{B}_\perp^T(x_c)T_c^{-T}(x_c)\mathcal{M}_a(P_c, x_c)T_c^{-1}(x_c)\tilde{B}_\perp(x_c) < 0, \tag{16}$$

$$\tilde{C}_\perp^T(x_c)\mathcal{M}_a(P_c, x_c)\tilde{C}_\perp(x_c) < 0 \tag{17}$$

for all $\tilde{B}_\perp(x_c)$ with SPAN $(\tilde{B}_\perp(x_c)) \in \mathcal{N}(\tilde{B}(x_c))$ and $\tilde{C}_\perp(x_c)$ with SPAN $(\tilde{C}_\perp(x_c)) \in \mathcal{N}(\tilde{C}(x_c))$.

Next, we consider (16), notice that $\mathcal{N}(\tilde{B}(x_c)) = \mathcal{N}(\breve{B}(x))$ for

$$\breve{B}(x) := \begin{bmatrix} B_2^T(x) & 0 & 0 & D_{12}^T(x) \\ 0 & I & 0 & 0 \end{bmatrix}.$$

Thence, (16) holds if and only if

$$\breve{B}_\perp^T(x)T_c^{-T}(x_c)\mathcal{M}_a(P_c, x_c)T_c^{-1}(x_c)\breve{B}_\perp(x) < 0 \tag{18}$$

for all $\breve{B}_\perp(x)$ with SPAN $(\breve{B}_\perp(x)) \in \mathcal{N}(\breve{B}(x))$. On the other hand, notice that (see (z) at the bottom of the next page).

Since $P_c(x_c) = P_c(x, \xi)$ is invertible on $X \times X_o$, assume $X(x) = X^T(x) \in R^{n \times n}$, which is positive definite and of class C^0 on X, is such that

$$P_c^{-1}(x, \phi(x)) = \begin{bmatrix} X(x) & X_1^T(x) \\ X_1(x) & X_0(x) \end{bmatrix} \tag{19}$$

for some continuously differentiable function $\phi\colon x \mapsto \xi$ in X such that $\phi(X) \subset X_0$ (for example, ϕ can be chosen as $\phi(x) = 0$). Therefore, by the arguments of Schur complements, (18), i.e., (16) implies (8). Thus, the first part is proved.

Next, consider (17), if we take $Y(x) \in R^{n \times n}$, which is of class C^0, such that

$$P_c(x, \phi(x)) = \begin{bmatrix} Y(x) & Y_1^T(x) \\ Y_1(x) & Y_0(x) \end{bmatrix}. \tag{20}$$

Notice that $\tilde{C}(x_c)$ just depends on $x \in X$, (17) implies (9).

Finally, by Lemma 7.2, (19) and (20) hold if and only if

$$\begin{bmatrix} X(x) & I \\ I & Y(x) \end{bmatrix} \geq 0.$$

This concludes the proof. □

Remark 4.2: It is noted that all couples $(X(x), Y(x))$ satisfying inequalities i), ii), and iii) form a convex set. Therefore, Theorem 4.1 provides a convex characterization to the necessary conditions for the strong output feedback $\mathcal{H}_\infty$-control problem to be solvable.

$$B_\perp^T(x)\begin{bmatrix} X(x)A^T(x) + A(x)X(x) + B_1(x)B_1^T(x) & X(x)C_1^T(x) + B_1(x)D_{11}^T(x) \\ C_1(x)X(x) + D_{11}(x)B_1^T(x) & D_{11}(x)D_{11}^T(x) - I \end{bmatrix}B_\perp(x) < 0 \tag{8}$$

$$C_\perp^T(x)\begin{bmatrix} A^T(x)Y(x) + Y(x)A(x) + C_1^T(x)C_1(x) & Y(x)B_1(x) + C_1^T(x)D_{11}(x) \\ B_1^T(x)Y(x) + D_{11}^T(x)C_1(x) & D_{11}^T(x)D_{11}(x) - I \end{bmatrix}C_\perp(x) < 0 \tag{9}$$

Remark 4.3: From the above proof, we can conclude that if the strong $\mathcal{H}_\infty$-control problem has a static output feedback solution, then there are two $\boldsymbol{C}^0$ positive definite matrix-valued functions $X, Y: \boldsymbol{X} \to \boldsymbol{R}^{n\times n}$ such that they satisfy conditions i) and ii) in Theorem 4.1, and $X(x)Y(x) = I$ for all $x \in \boldsymbol{X}$. Note that the characterization is not convex in this case.

It is noted that in general, the NLMI characterization in Theorem 4.1 is not sufficient, because on one hand, the strong $\mathcal{H}_\infty$-control characterization by NLMI (15) holds only if (16) and (17) hold, and the converse implication in general is not true since the matrix-valued matrix function $F_c(x_c)$ has some special structure (13) which is not guaranteed to recover by Lemma 7.3; on the other hand, as noted in Remark 2.4, the existence of the positive definite matrix-valued function $P_c(x_c)$ satisfying the NLMI (14) is not enough to guarantee the closed-loop system has storage function $V_c: \boldsymbol{X} \times \boldsymbol{X}_o \to \boldsymbol{R}^+$ such that $\partial V_c/\partial x_c(x_c) = 2x^T P_c(x_c)$, some additional constraints are required (see Proposition 5.2). Nonetheless, the characterization is sufficient if the $\mathcal{H}_\infty$-control problem is considered locally as states as follows.

Theorem 4.4: Consider system (6), there exists a local solution to the output feedback (strong) $\mathcal{H}_\infty$ control problem if and only if there are two $\boldsymbol{C}^0$ positive definite matrix-valued functions $X, Y: \mathcal{B}_r \to \boldsymbol{R}^{n\times n}$ with $\mathcal{B}_r \subset \boldsymbol{R}^n$ for some $r > 0$ such that they satisfy the conditions i), ii), and iii) in Theorem 4.1 for all $x \in \mathcal{B}_r$.

Proof: The necessity follows from the previous theorem. The sufficiency follows from the continuity argument, we just give an outline for this part. We make a simplification assumption that $D_{22}(x) = 0$ without loss of generality.

Suppose two $\boldsymbol{C}^0$ positive definite matrix-valued functions $X, Y: \mathcal{B}_r \to \boldsymbol{R}^{n\times n}$ satisfy i), ii), and iii) for all $x \in \mathcal{B}_r$. By the continuity of the coefficient matrix-valued functions A, B_i, C_j, D_{ij} $(i, j = 1, 2)$, $B_\perp$, and $C_\perp$, it follows that the three NLMI's have local constant positive definite solutions $X_c := X(0)$, $Y_c := Y(0)$ for all $x \in \mathcal{B}_r$ with some adjusted $r > 0$. From condition iii) and Lemma 7.2, it follows that we can find a constant positive definite matrix P_c such that

$$P_c^{-1} = \begin{bmatrix} X_c & X_1^T \\ X_1 & X_0 \end{bmatrix}, \quad P_c = \begin{bmatrix} Y_c & Y_1^T \\ Y_1 & Y_0 \end{bmatrix}.$$

Moreover, the constant matrix P_c locally satisfies (16) and (17). Now by Lemma 7.3, P_c locally satisfies (14) with the coefficient matrix-valued functions defined by (11) and (12) for some matrix valued function $F_c(x_c)$. Still by continuity argument, we can find a constant matrices $\hat{A}, \hat{B}, \hat{C}$, and $\hat{D}$, such that (14) locally holds by replacing $F_c(x_c)$ with $\hat{F}_c := \begin{bmatrix} \hat{D} & \hat{C} \\ \hat{B} & \hat{A} \end{bmatrix} := F_c(0)$. Define a positive definite function $V_c(x_c) := x_c^T P_c x_c$ with $x_c := \begin{bmatrix} x \\ \xi \end{bmatrix}$, then $\partial V_c/\partial x_c(x_c) = 2x_c^T P_c$. Therefore, by Theorem 2.2, the controller given by

$$K_c: \begin{cases} \dot{\xi} = \hat{A}\xi + \hat{B}y \\ u = \hat{C}\xi + \hat{D}y \end{cases}$$

is a local strong $\mathcal{H}_\infty$-controller, and the resulting closed-loop system is locally asymptotically stable. □

It is noted that in the above solutions to the strong $\mathcal{H}_\infty$-control problem for system (6), the $\mathcal{H}_\infty$-controllers are not required to have separation structures. The implications of NLMI's (8) and (9) in the last section have not been sufficiently revealed. This issue will be pursued further next. Actually, the NLMI's (8) and (9) are closely related to the state-feedback and output-injection conditions for nonlinear $\mathcal{H}_\infty$-control. It will be shown that under a weaker separation structure constraints, if the $\mathcal{H}_\infty$-control problem is solvable by output feedback, then it is also solvable by static-state feedback.

We first state a theorem which justifies that NLMI (8) characterizes state-feedback solution under additional constraints.

Theorem 4.5: The strong $\mathcal{H}_\infty$-control problem is solvable by static state feedback if and only if there is a $\boldsymbol{C}^0$ matrix-valued function $X(x) = X^T(x) > 0$ with $\partial V/\partial x(x) = 2x^T X^{-1}(x)$ for some $\boldsymbol{C}^1$ function $V: \boldsymbol{X} \to \boldsymbol{R}^+$ such that for all $x \in \boldsymbol{X}$, the following NLMI holds (as shown by (21) at the bottom of the next page) with $B_\perp: \boldsymbol{X} \to \boldsymbol{R}^{(n+q_1)\times(n+q_1-p_2)}$ such that $\mathrm{SPAN}\,(B_\perp(x)) = \mathcal{N}(B(x))$.

Proof: The necessity basically follows the arguments in the proof of Theorem 4.1. The sufficiency also follows the proof of Theorem 4.1 by noting the converse direction in the proof goes through in this case, since a smooth static-state feedback can be constructed by using Lemma 7.3; then the conclusion follows by Theorem 2.2. □

Next, we will find the relation between output feedback solutions and state-feedback solutions. Suppose the output feedback strong $\mathcal{H}_\infty$-control problem for the given system (6) is solvable, then there is a $\boldsymbol{C}^0$ positive definite matrix-valued function $P_c(x_c)$ such that (14) holds. Moreover, there is a positive definite function $V_c: \boldsymbol{X} \times \boldsymbol{X}_0 \to \boldsymbol{R}^+$ such that

$$\frac{\partial V_c}{\partial x_c}(x_c) = 2x_c^T P_c(x_c). \tag{22}$$

Assumption 4.6: Consider the positive definite function $V_c: \boldsymbol{X} \times \boldsymbol{X}_0 \to \boldsymbol{R}^+$ satisfying (22). There is a $\boldsymbol{C}^1$ function $\phi: x \mapsto \xi$ with $\phi(0) = 0$ such that $\partial V_c/\partial \xi(x, \xi)|_{\xi=\phi(x)} = 0$ with $(x, \xi) \in \boldsymbol{X}\times\boldsymbol{X}_0$.

Remark 4.7: Note that the function V_c is a Lyapunov function of the closed-loop system. Assumption 4.6 has a (weak) separation structure interpretation. In fact, many dynamical controllers have well-defined separation structures [1], [10], [14]. In such a case, the states x, ξ of a plant and its controller satisfy that

$$\xi(t) - \phi(x(t)) \to 0 \text{ as } t \to \infty$$

for some $\boldsymbol{C}^1$ function $\phi: x \mapsto \xi$ with $\phi(0) = 0$; in particular, if the initial state satisfy $\xi(0) = \phi(x(0))$, then $\xi(t) = \phi(x(t))$ for all $t \in \boldsymbol{R}^+$. A Lyapunov function U_c for the closed-loop system is constructed as follows

$$U_c(x, \xi) = V(x) + U(\xi - \phi(x))$$

$$\mathcal{M}_a(P_c, x_c) := \begin{bmatrix} (A^a(x))^T P_c(x_c) + P_c(x_c)A^a(x) & P_c(x_c)B_1^a(x) & (C_1^a(x))^T \\ (B_1^a(x))^T P_c(x_c) & -I & (D_{11}^a(x))^T \\ C_1^a(x) & D_{11}^a(x) & -I \end{bmatrix}$$
$$\tilde{B}(x_c) := [(B_2^a(x_c))^T \quad 0 \quad (D_{12}^a(x))^T], \qquad \tilde{C}(x_c) := [C_2^a(x) \quad D_{21}^a(x) \quad 0] \tag{y}$$

$$T_c^{-T}(x_c)\mathcal{M}_a(P_c, x_c)T_c^{-1}(x_c) = \begin{bmatrix} P_c^{-1}(x_c)(A^a(x))^T + A^a(x)P_c^{-1}(x_c) & B_1^a(x) & P_c^{-1}(x)(C_{11}^a(x))^T \\ (B_1^a(x))^T & -I & D_{11}^T(x) \\ C_1^a(x)P_c^{-1}(x_c) & D_{11}(x) & -I \end{bmatrix} \tag{z}$$

where V and U are Lyapunov functions of the state-feedback system and the error system. Thence

$$\frac{\partial U_c}{\partial \xi}(x, \xi) = \frac{\partial U}{\partial e}(e)\bigg|_{e=\xi-\phi(x)}.$$

If $e = 0$, i.e., $\xi = \phi(x)$, then

$$\frac{\partial U_c}{\partial \xi}(x, \xi)\bigg|_{\xi=\phi(x)} = \frac{\partial U}{\partial e}(e)\bigg|_{e=0} = 0.$$

Therefore, U_c satisfies the assumption.

From the proof of Theorem 4.1, it follows that (14) implies that there is a C^0 positive definite matrix-valued function $X: \boldsymbol{X} \to \boldsymbol{R}^{n\times n}$ such that

$$P_c^{-1}(x, \phi(x)) = \begin{bmatrix} X(x) & X_1^T(x) \\ X_1(x) & X_0(x) \end{bmatrix}$$

for some continuously differentiable function $\phi: x \mapsto \xi$ on $\boldsymbol{X}$, and NLMI (21) holds. Since $\partial V_c/\partial x_c(x_c) = 2x_c^T P_c(x_c)$ implies $\partial V_c/\partial x_c(x_c)P_c^{-1}(x_c) = 2x_c^T$, or

$$\left[\frac{\partial V_c}{\partial x}(x, \xi) \quad \frac{\partial V_c}{\partial \xi}(x, \xi)\right] P_c^{-1}(x, \xi) = 2[x^T \quad \xi^T]. \tag{23}$$

Take the function ϕ as in Assumption 4.6, define $V(x) := V_c(x, \phi(x))$, then (23) implies

$$\frac{\partial V_c}{\partial x}(x, \phi(x))X(x) = 2x^T.$$

Define $V(x) := V_c(x, \phi(x))$, then $V(x)$ is positive definite, and by Assumption 4.6

$$\frac{\partial V}{\partial x}(x) = 2x^T X^{-1}(x). \tag{24}$$

Therefore, the $\mathcal{H}_\infty$-control problem is solvable in terms of static feedback. The above observation is summarized as follows.

Theorem 4.8: If the strong $\mathcal{H}_\infty$-control problem is solvable (by the output feedback), and the corresponding Lyapunov function satisfies Assumption 4.6, then the problem can also be solved by static state feedback.

Similar argument applies to output injection problem [15].

V. Computational Implications of Nonlinear Matrix Inequalities

In this section, we will address computational issue for strong $\mathcal{H}_\infty$-control performance analysis and synthesis. We have known that, the $\mathcal{H}_\infty$-control performance analysis and synthesis involves solving some NLMI's, i.e., (3), (8), (9), which result convex problems. This property also implies that the computational effort needed in $\mathcal{H}_\infty$-performance analysis and $\mathcal{H}_\infty$-control is not more difficult than that for checking Lyapunov stability [16]. In this section, we will examine some other properties of NLMI's related to the solutions to $\mathcal{H}_\infty$-control problems.

A. Existence of Continuous Solutions

The solvability for each strong $\mathcal{H}_\infty$-control problem requires that the positive definite solutions to the corresponding NLMI's be continuous; in this subsection, we will show that if an NLMI has a pointwise positive definite solution, then there exists a continuous one.

Let $\boldsymbol{X}$ be an open subset $\boldsymbol{R}^n$ with $0 \in \boldsymbol{X}$, consider a general matrix-valued map $\mathcal{M}: \boldsymbol{R}^{n\times n} \times \boldsymbol{X} \to \boldsymbol{R}^{m\times m}$, which is continuous and satisfies

$$\mathcal{M}\left(\sum_{k=1}^{N} \alpha_k P_k, x\right) = \sum_{k=1}^{N} \alpha_k \mathcal{M}(P_k, x) \tag{25}$$

for all $\alpha_k \geq 0$ with $\sum_{k=1}^N \alpha_k = 1$. Considered the following matrix inequality

$$\mathcal{M}(P, x) < 0. \tag{26}$$

Note that all of the NLMI's discussed in this paper are in this matrix inequality class.

The main result of this subsection is stated as follows; the proof employs the partitions of unity arguments and is given in [15] and [16].

Theorem 5.1: Suppose the matrix inequality (26) has a positive definite solution P_x for each $x \in \boldsymbol{X}$, i.e., $\mathcal{M}(P_x, x) < 0$ for $x \in \boldsymbol{X}$, then there exists a C^0 (in fact, C^∞) positive-definite matrix-valued function $P: \boldsymbol{X} \to \boldsymbol{R}^{n\times n}$, such that $\mathcal{M}(P(x), x) < 0$ for all $x \in \boldsymbol{X}$.

B. Existence of Lyapunov Functions

As mentioned in Remark 2.4, the existence of positive definite matrix-valued function $P: \boldsymbol{X} \to \boldsymbol{R}^{n\times n}$ to NLMI's is not enough to guarantee the strong $\mathcal{H}_\infty$-control problem to have solution; some additional requirement is required, i.e., there is a C^1 storage function, $V: \boldsymbol{X} \to \boldsymbol{R}^+$, such that

$$\frac{\partial V}{\partial x}(x) = 2x^T P(x)$$

for all $x \in \boldsymbol{X}$. The following result is quite standard (see, for example, [2, Lemma 2.22]).

Proposition 5.2: Suppose a vector-valued function $p: \boldsymbol{X} \to \boldsymbol{R}^n$ is of class C^k for some integer $k \geq 1$; let

$$p(x) = [p_1(x), \cdots, p_n(x)]^T \text{ for } x \in \boldsymbol{X}.$$

Then there exists a C^{k+1} function $V: \boldsymbol{X} \to \boldsymbol{R}$ such that

$$\frac{\partial V}{\partial x}(x) = 2p^T(x)$$

if and only if

$$\frac{\partial p_i}{\partial x_j}(x) = \frac{\partial p_j}{\partial x_i}(x) \tag{27}$$

for all $x \in \boldsymbol{X}$ and $i, j = 1, 2, \cdots, n$. Moreover, if (27) holds, then an function $V: \boldsymbol{X} \to \boldsymbol{R}$ with $V(0) = 0$ is given by

$$V(x) = 2x^T \int_0^1 p(tx)\, dt. \tag{28}$$

In addition, if $p(x) = P(x)x$ for some C^k positive definite matrix-valued function $P: \boldsymbol{X} \to \boldsymbol{R}^{n\times n}$, then $V(x)$ is also positive definite function.

$$B_\perp^T(x)\begin{bmatrix} X(x)A^T(x) + A(x)X(x) + B_1(x)B_1^T(x) & X(x)C_1^T(x) + B_1(x)D_{11}^T(x) \\ C_1(x)X(x) + D_{11}(x)B_1^T(x) & D_{11}(x)D_{11}^T(x) - I \end{bmatrix} B_\perp(x) < 0 \tag{21}$$

C. Existence of (Local) Constant Solutions to NLMI's

The above treatments about $\mathcal{H}_\infty$-performance analysis and synthesis are in terms of NLMI's, which are pointwise LMI's on state set $\boldsymbol{X}$, modulo some additional constraints on the solutions. We also know that if set $\boldsymbol{X}$ is small enough, then we can get a constant solution to the NLMI's. Next, we will use a similar treatment to that used in [3], which is motivated by the notion of global linearization of nonlinear systems developed by Liu *et al.* [12]. More concretely, we consider the following NLMI

$$\begin{bmatrix} A^T(x)P(x)+P(x)A(x) & P(x)B(x) & C^T(x) \\ B^T(x)P(x) & -I & D^T(x) \\ C(x) & D(x) & -I \end{bmatrix} < 0$$

where the coefficient matrix-valued functions $A(x)$, $B(x)$, $C(x)$, $D(x)$ are assumed to be continuous on $\boldsymbol{X}$. The coefficient matrices are assumed in a convex set, i.e.,

$$[A(x), B(x), C(x), D(x)] \in \mathrm{Co}\Big\{[A_i, B_i, C_i, D_i]|_{i\in\{1,2,\cdots,L\}}\Big\}, \quad \forall x \in \boldsymbol{X}$$

for some A_i, B_i, C_i, D_i with $I - D_i^T D_i \le 0$ for $i \in \{1, 2, \cdots, L\}$ with some integer $L > 0$, where Co stands for the convex hull.

If there is a constant (semi-)positive definite matrix $P \in \boldsymbol{R}^{n\times}$ such that

$$\begin{bmatrix} A_i^T P + PA_i & PB_i & C^T \\ B_i^T P & -I & D_i^T \\ C_i & D_I & -I \end{bmatrix} < 0, \quad \forall i \in \{1, 2, \cdots, L\}$$

which are a set of LMI's and can be solved in terms of convex optimization methods [3], then P also satisfies

$$\begin{bmatrix} A^T(x)P+PA(x) & PB(x) & C^T(x) \\ B^T(x)P & -I & D^T(x) \\ C(x) & D(x) & -I \end{bmatrix} < 0.$$

for all $x \in \boldsymbol{X}$.

The solution automatically satisfies condition (27), and the corresponding Lyapunov function is $V(x) = x^T Px$.

This treatment suggests a tractable approach to get local solutions. This approach however, generally leads to conservative results if the prescribed state set is too large.

VI. Concluding Remarks

In this paper, the $\mathcal{H}_\infty$-control problem for a class of nonlinear systems has been characterized in terms of nonlinear matrix inequalities which result in the convex problems. This implies that the computation needed for $\mathcal{H}_\infty$-control is not more difficult than that for checking Lyapunov stability. Unfortunately, unlike the linear case, the solution of the NLMI's by themselves are not sufficient to guarantee the existence of the required $\mathcal{H}_\infty$-controller. The proposed approach, however, points out a new direction to make the nonlinear $\mathcal{H}_\infty$-control theory to be applicable.

VII. Appendix
Some Technical Results

A. Schur Complements

A reference for the material here is [8].

Lemma 7.1: Suppose $M = M^T \in \boldsymbol{R}^{(n+m)\times(n+m)}$ is partitioned as

$$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

with $C \in \boldsymbol{R}^{m+m}$ is nonsingular, then $M \ge 0$ if and only if $C > 0$ and $A - BC^{-1}B^T \ge 0$.

Lemma 7.2: Let $X = X^T$, $Y = Y^T \in \boldsymbol{R}^{n\times n}$ be two positive definite matrices. Then there is a positive definite matrix $P = P^T \in \boldsymbol{R}^{(n+m)\times(n+m)}$ such that

$$P = \begin{bmatrix} X & X_1^T \\ X_1 & X_0 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} Y & Y_1^T \\ Y_1 & Y_0 \end{bmatrix}$$

if and only if $\left[\begin{smallmatrix} X & I \\ I & Y \end{smallmatrix}\right] \ge 0$.

B. Characterization of a State-Dependent LMI

Given an integer $k \ge 0$ and a $\boldsymbol{C}^k$ matrix-valued function $B: \boldsymbol{X} \to \boldsymbol{R}^{m\times n}$, with $m \le n$, and $\mathrm{RANK}\,(B(x)) = m$ for all $x \in \boldsymbol{X}$. Thus, there is an $(n-m)$-dimensional distribution $\mathcal{N}(B(x))$ on $\boldsymbol{X}$ which annihilates the row vectors in $B(x)$. Moreover, there is a $\boldsymbol{C}^k$ matrix-valued function $B_\perp: \boldsymbol{M} \to \boldsymbol{R}^{n\times(n-m)}$, such that its columns span the distribution $\mathcal{N}(B(x))$, i.e., $\mathcal{N}(B(x)) = \mathrm{SPAN}\,(B_\perp(x))$ for $x \in \mathcal{M}$. The following lemma generalizes Finsler's Theorem (see, for example, [4] and [3]).

Lemma 7.3: Given three $\boldsymbol{C}^k$ matrix-valued functions $Q = Q^T: \boldsymbol{X} \to \boldsymbol{R}^{m\times m}$, $U: \boldsymbol{X} \to \boldsymbol{R}^{r\times m}$ with $\mathrm{RANK}\,(U(x)) = r < m$, and $V: \boldsymbol{X} \to \boldsymbol{R}^{s\times m}$ with $\mathrm{RANK}\,(V(x)) = s < m$. There exists a $\boldsymbol{C}^k$ matrix-valued function $F: \boldsymbol{X} \to \boldsymbol{R}^{s\times r}$ such that the following matrix inequality is satisfied

$$Q(x) + U^T(x)F^T(x)V(x) + V^T(x)F(x)U(x) < 0 \qquad (29)$$

if and only if

$$U_\perp^T(x)Q(x)U_\perp(x) < 0, \quad V_\perp^T(x)Q(x)V_\perp(x) < 0 \qquad (30)$$

for some $\boldsymbol{C}^k$ matrix-valued functions $U_\perp: \boldsymbol{X} \to \boldsymbol{R}^{m\times(m-r)}$ with $\mathrm{SPAN}\,(U_\perp(x)) = \mathcal{N}(U(x))$ and $V_\perp: \boldsymbol{X} \to \boldsymbol{R}^{m\times(m-s)}$ with $\mathrm{SPAN}\,(V_\perp(x)) = \mathcal{N}(V(x))$.

Proof: The necessity is obvious. As for the sufficiency, suppose condition (30) is satisfied. From the constant matrix version of the above lemma, it follows that for each fixed $x \in \boldsymbol{X}$, there exists a matrix $F_x \in \boldsymbol{R}^{s\times r}$ such that

$$Q(x) + U^T(x)F_x^T V(x) + V^T(x)F_x U(x) < 0$$

i.e., NLMI (29) has a pointwise solution. Using the partition of unity argument, we can find a smooth matrix-valued function $F: \boldsymbol{X} \to \boldsymbol{R}^{s\times r}$ such that NLMI (29) is satisfied. □

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On Sufficient Conditions for Stability Independent of Delay

Jie Chen, Demin Xu, and Bahram Shafai

***Abstract*— In this note we study the stability properties of linear time-invariant delay systems. The specific notion under consideration is asymptotic stability independent of delay. As an attempt to achieve a compromise between the complexity and tightness of various stability tests, we present a number of sufficient stability conditions which improve several previously available sufficient conditions and which are also much easier to verify than the known necessary and sufficient conditions.**

I. Introduction

In this note we study stability properties for mainly the class of linear time-invariant delay systems described by the differential-difference equation

$$\dot{x}(t) = Ax(t) + Bx(t-h), \quad h \geq 0 \tag{1.1}$$

where $A, B \in \mathbf{R}^{n\times n}$ are known system matrices, and h is a delay constant. A specific notion under consideration is asymptotic stability independent of delay [17]. Roughly, we say that system (1.1) is asymptotically stable independent of delay if it is stable with respect to all delay constants $h \geq 0$. Alternatively, system (1.1) is asymptotically stable independent of delay if and only if it satisfies the condition

$$\det\left(sI - A - Be^{-hs}\right) \neq 0, \quad \forall\, s \in \overline{\mathbf{C}}_+, \quad \forall\, h \geq 0 \tag{1.2}$$

where $\mathbf{C}_+ := \{s\colon \mathrm{Re}(s) > 0\}$ denotes the open right-half plane, and $\overline{\mathbf{C}}_+$ denotes its closure. This condition simply states that the characteristic polynomial of (1.1) has no roots in the closed right-half plane, with respect to all nonnegative delay values.

Stability of time-delay systems has been a much studied problem (see, e.g., [1], [11], [19], [20], and the references therein). Numerous results have been developed concerning specifically the notion of stability independent of delay for linear time-invariant delay systems either described by (1.1) or in a more general form. Results that are most closely related to this note are necessary and sufficient conditions obtained in [12] and [6] and sufficient conditions developed in [3], [14], [21], [22], [24], [30], [31]. The stability test in [12] is based upon stability properties of two-variable polynomials and in general may be difficult to verify. The more recent result of [6], though may be significantly simpler, also requires checking a frequency dependent measure. In contrast, the sufficient conditions reported in [21], [22], [24], [31] require checking only certain constant measures and therefore are considerably less intensive in computation. Additional related results may also be found in, e.g., [5], [10], [13], [15]–[18], [23].

The purpose of this note is to present a number of additional sufficient conditions which may be used to test the stability of (1.1) independent of delay and which should complement the aforementioned results. These conditions can be verified more easily than those in [5], [6], [12], [15], and [16], and are less conservative than those of [21], [22], [24], and [31]. In deriving these results, our emphasis has been to compromise between the complexity and tightness of various stability tests, and we feel that intermediate results of this sort should be useful. Our derivation is based on the necessary and sufficient condition of [6]. Section II contains our main results. We first present a sufficient condition which amounts to checking the norms of a frequency dependent matrix. This result is similar to conditions given in [10], [18], [30], and [32] but is more general. The verification of this condition can be made simpler than that required in [6] by a proper choice of norms. Next, we give a slightly weakened result which requires solving a Lyapunov matrix equation. Both these results are shown to be less conservative than those of [21], [22], [24], [31]. In Section III, these results are further extended to a more general class of systems which contain multiple noncommensurate delays. Section IV concludes our discussion.

A preliminary version of this paper was previously presented in [7].

II. Main Results

In the sequel, the symbol $\lambda_i(\cdot)$ denotes the ith eigenvalue of a matrix and $\rho(\cdot)$ denotes its spectral radius. For a Hermitian matrix, $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denote, respectively, its largest and smallest eigenvalues. The superscripts T and H denote matrix transpose

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