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ON INNER-OUTER AND SPECTRAL FACTORIZATIONS

Cheng-Chih Chu   John C. Doyle

Honeywell Inc., Systems and Research Center, Minneapolis, Minnesota

Abstract

This paper outlines methods for computing the key factorizations necessary to solve general H_2 and H_∞ linear optimal control problems.

Notation

$L_\alpha \triangleq$ { Lebesque space }

$H_\alpha \triangleq$ { Hardy space }

$R \triangleq$ { Proper, real-rational }

$R^{p\times m} \triangleq$ {$p\times m$ matrices in R} (similarly for H and L)

$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \triangleq D + C(sI-A)^{-1}B$

$rcf \triangleq$ right coprime factorization over RH_∞

Throughout this paper, α will be used whenever either α=2 or $\alpha=\infty$ would apply equally. The term *unit* in RH_∞ refers to any $M \in RH_\infty$ such that $M^{-1} \in RH_\infty$. When R is used as a prefix, it denotes real-rational.

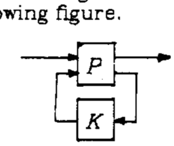
1. Introduction :

The importance of inner-outer (**IOF**), spectral and coprime factorizations in obtaining solutions to certain H_2 and H_∞ optimal control problems has been known for some time. The solution to the general H_α ($\alpha=2,\infty$) optimal control problems [1],[2] uses these factorizations and,in addition, the "complementary inner factor" (**CIF**), to reduce the general problem to that of approximating an L_α rational matrix by one in H_α.

This paper focuses on the factorizations used in [2] and, in particular, on explicit formulas and methods for computation. We show that all the factorizations needed in the H_2 and H_∞ optimal synthesis problem can be obtained using standard real matrix operations on state-space representations. The Algebraic Riccati Equation (**ARE**) plays a central role in computing the desired factorizations. Because of space limitations, the "proofs" of the results in this paper are extremely sketchy.

2. Background :

The general H_α optimal control problem is shown in the following figure.



$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in R^{(p_1+p_2)\times(m_1+m_2)}$$

The objective is to find a stabilizing $K \in R^{m_2\times p_2}$ which solves $\min_K ||\mathbf{F}_l(P;K)||_\alpha$ where $\mathbf{F}_l(P;K) \triangleq P_{11}+P_{12}K(I-P_{22}K)^{-1}P_{21}$. For nontriviality, assume that $p_1 > m_2$ and $m_1 > p_2$.

The first step is to find $K_0 = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \in R^{(m_2+p_2)\times(p_2+m_2)}$ such that $\mathbf{F}_l(P;\mathbf{F}_l(K_0;Q)) = \mathbf{F}_l(T;Q) = T_{11}+NQ\tilde{N} \in RH_\alpha^{p_1\times m_1}$ is stable and affine for any $Q \in RH_\infty^{m_2\times p_2}$. This is the Youla parametrization of all stabilizing controllers and is obtained by finding coprime factorizations of P over the ring of stable rationals and solving a double Bezout identity to obtain the coefficients of K_0.

We are interested in a particular K_0 which results in both N and $\tilde{N}$ being inner. That is, $N^*N=I$ and $\tilde{N}\tilde{N}^*=I$. This requires a coprime factorization with inner numerator. In addition, we require $N_\perp$ and $\tilde{N}_\perp$ inner so that $\begin{bmatrix} N & N_\perp \end{bmatrix}$ and $\begin{bmatrix} \tilde{N} \\ \tilde{N}_\perp \end{bmatrix}$ are *square* and inner. $N_\perp$ and $\tilde{N}_\perp$ are called complementary inner factors (**CIF**). With these we have that

$$\left\|T_{11}+NQ\tilde{N}\right\|_\alpha = \left\|\begin{bmatrix} N & N_\perp \end{bmatrix}^*[T_{11}+NQ\tilde{N}]\begin{bmatrix} \tilde{N} \\ \tilde{N}_\perp \end{bmatrix}^*\right\|_\alpha$$

$$= \left\|\begin{bmatrix} N & N_\perp \end{bmatrix}^*[T_{11}]\begin{bmatrix} \tilde{N} \\ \tilde{N}_\perp \end{bmatrix}^* + \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}\right\|_\alpha \qquad (2.1)$$

since both the α = 2 and ∞ norms are unitary invariant.

The H_2 case immediately reduces to a best approximation problem with $Q_{opt} = \mathbf{P}_{H_2}(N^*[T_{11}]\tilde{N}^*)$, where $\mathbf{P}_{H_2}$ denotes projection onto H_2. The H_∞ case is somewhat more complicated and requires an additional spectral factorization. To see how this arises, consider the special case when $T_{11}\tilde{N}_\perp{}^* = 0$ and (2.1) reduces to $\left\|\begin{bmatrix} R+Q \\ G \end{bmatrix}\right\|_\infty$ with $R = N^*[T_{11}]\tilde{N}^*$ and $G = N_\perp^*[T_{11}]\tilde{N}^*$.

It is easily verified that for any $\gamma > ||G||_\infty$

$$\left\|\begin{bmatrix} R+Q \\ G \end{bmatrix}\right\|_\infty \le \gamma \quad iff \quad \left\|\left(\gamma^2 I - G^*G\right)^{-1/2}\left(R+Q\right)\right\|_\infty \le 1$$

where $(H)^{1/2}$ denotes the unit spectral factor of the para-Hermitian matrix H. Thus, the H_∞ problem also reduces to a best approximation problem since the $\left(\gamma^2 I - G^*G\right)^{-1/2}$ is a unit and can be absorbed into Q. The general case similarly involves both inner-outer and spectral factorizations [2]. The remainder of the paper outlines methods for computing these factorizations.

3. Algebraic Riccati Equation :

Consider the Algebraic Riccati Equation,

$$F^TX + XF - XWX + Q = 0 \qquad \text{(ARE)}$$

where $F, W, Q \in \mathbb{R}^{n\times n}$, $W = W^T \ge 0$ and $Q = Q^T$ with the associated Hamiltonian matrix

$$A_H = \begin{bmatrix} F & -W \\ -Q & -F^T \end{bmatrix}$$

Our main interest is to find the unique real symmetric stabilizing solution such that the matrix $(F - WX)$ is asymptotically stable. For simplicity we will use "solution" of the **ARE** to mean a real symmetric one. The **ARE** considered here is more general than the **ARE** which arises in linear quadratic optimal control and Kalman-Bucy filtering theory in that there is no assumption on the definiteness of the matrix Q.

The following theorem gives the necessary and sufficient conditions for the existence of a unique stabilizing solution of (**ARE**). Without loss of generality, we will assume that $W = GG^T$. This is a slight generalization of a theorem of Kucera [3].

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Theorem 3-1 :

The stabilizability of (F,G) and $\mathrm{Re}[\lambda_i(A_H)] \neq 0$ ($\forall\ i = 1,2,\ldots,2n$) are necessary as well as sufficient for the existence of a unique stabilizing solution of **(ARE)**.

Remarks :

(1) The unique stabilizing solution of Theorem 3-1 will be denoted by **Ric** (A_H).

(2) If $Q \geq 0$, then the stabilizing solution $X \geq 0$.

Theorem 3-2 :

If $Q = H^T H \geq 0$ in **(ARE)** and X is its solution, then $Ker(X) \subset Ker(H)$.

4. Inner-Outer Factorization (IOF) :

Assume $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in R^{p\times m}$ $(p \geq m)$ and the realization is minimal. We use the notation "$D_\perp$" for the orthogonal complement of D and $R^{½}$ $(R \geq 0)$ for any square matrix V s.t. $V^T V = R$.

Theorem 4-1 :

G has a *rcf* $G = NM^{-1}$ with N inner if and only if G has no (transmission) zeros on the $j\omega$-axis, including ∞.

A particular realization for the factorization is

$$\begin{bmatrix} M \\ N \end{bmatrix} = \left[\begin{array}{c|c} A-BF & BR^{-½} \\ \hline -F & R^{-½} \\ C-DF & DR^{-½} \end{array}\right] \in RH_\infty^{(m+p)\times m}$$

where $R = D^T D > 0$, $F = R^{-1}(B^T X - D^T C)$,

and $X = \mathbf{Ric}\begin{bmatrix} A-BR^{-1}D^TC & -BR^{-1}B^T \\ -C^T D_\perp D_\perp^T C & -(A-BR^{-1}D^TC)^T \end{bmatrix} \geq 0.$

[Proof] :

(only if) :

Suppose $G = NM^{-1}$ is a *rcf* and $N^*N = I$. Then $G^*G = (M^{-1})^* M^{-1} > 0$ on the $j\omega$-axis since M is stable. Thus G has no $j\omega$-axis zeros.

(if) :

The if part can be proven by directly verifying that the above realization is a *rcf* of G and that $N^*N = I$. The conditions on G insure, by Theorem 3-1, the existence of X.

Theorem 4-2 :

If $p > m$ in Theorem 4-1, then there exists a **CIF** $N_\perp \in RH_\infty^{p\times(p-m)}$ such that the matrix $[\ N\ N_\perp\ ]$ is square and inner. A particular realization is

$$N_\perp = \left[\begin{array}{c|c} A-BF & -X^\dagger C^T D_\perp \\ \hline C-DF & D_\perp \end{array}\right]$$

where $X^\dagger$ is the pseudo-inverse of the stabilizing Riccati solution X from Theorem 4-1.

[Proof] :

The proof consists of verifying directly that $[\ N\ N_\perp\ ]$ is inner using the above realization for $N_\perp$ and the realization for N from Theorem 4-1. A key step requires that $Ker(X) \subset Ker(D_\perp^T C)$, which follows from Theorem 3-2.

Theorem 4-3 :

There exists a *rcf* $G = NM^{-1}$ such that M $\in RH_\infty^{m\times m}$ is inner if and only if G has no poles on the $j\omega$-axis. A particular realization is

$$\begin{bmatrix} M \\ N \end{bmatrix} = \left[\begin{array}{c|c} A-BF & B \\ \hline -F & I \\ C-DF & D \end{array}\right] \in RH_\infty^{(m+p)\times m}$$

where $F = B^T X$ and $X = \mathbf{Ric}\begin{bmatrix} A & -BB^T \\ 0 & -A^T \end{bmatrix} \geq 0.$

[Proof] :

Similar to Theorem 4-1.

Remarks :

(1) The minimality condition in Theorem 4-3 can be weakened to (A,B) stabilizable and $\mathrm{Re}[\lambda_i(A)] \neq 0$ and the theorem still holds.

(2) If $G \in RH_\infty^{p\times m}$ in Theorem 4-1, then M is a unit in RH_∞. In this case, $G = N(M^{-1})$ is called "inner-outer factorization" **(IOF)**.

(3) Dual results for all factorizations can be obtained when $p \leq m$. In these factorizations, output injection using the dual Riccati solution replaces state feedback to obtain corresponding left factorizations.

5. Spectral Factorization :

Recall from Section 2 that the H_∞ solution requires the computation of the spectral factor which is a unit in RH_∞. The following theorem characterizes this spectral factor. The same assumptions on G apply here as did in Section 4 except that $G \in RL_\infty^{p\times m}$.

Theorem 5-1 :

If $\gamma > \|G(s)\|_\infty$, there exists a unit $M \in RH_\infty^{m\times m}$ such that $M^*M = \gamma^2 I - G^*G$. A particular realization is given by

$$M = \left[\begin{array}{c|c} A-K_F K_C & B-K_F \\ \hline R^{½}K_C & R^{½} \end{array}\right]$$

where $R = \gamma^2 I - D^T D > 0$, $K_C = R^{-1}(B^T X - D^T C)$, $K_F = Y K_C^T R$,

$$X = \mathbf{Ric}\begin{bmatrix} A+BR^{-1}D^TC & -BR^{-1}B^T \\ C^T(I+DR^{-1}D^T)C & -(A+BR^{-1}D^TC)^T \end{bmatrix}$$

and

$$Y = \mathbf{Ric}\begin{bmatrix} A^T & -K_C^T R K_C \\ 0 & -A \end{bmatrix}$$

[Proof] :

The proof consists of verifying from the above realizations that M and M^{-1} are stable and that $M^*M = \gamma^2 I - G^*G$.

Remark :

If G is stable, then $Y = 0$ and $K_F = 0$.

6. Implementation :

All the factorizations in the theorems have explicit state-space representations. Thus, the implementation of algorithms can be done easily using real matrix operations. Two key steps in the algorithms are:

(1) Solution of **ARE** : the **ARE** can be solved using the Schur method which reduces the Hamiltonian matrix to real Schur form by orthogonal similarity transformation and then uses the invariant subspace associated with n stable eigenvalues to find the stabilizing solution. See [4] for a complete treatment.

(2) $(D^T D)^{½}$ and $D_\perp$: using the QR factorization, we find $D = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}\begin{bmatrix} R \\ 0 \end{bmatrix}$ where $Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$ is orthogonal. Then $R = (D^T D)^{½}$ and $D_\perp = Q_2$.

Using the well-known software packages EISPACK, LINPACK and some of their extensions, experimental versions of these algorithms have been implemented successfully. Experience to date indicates that these algorithms are very reliable.

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