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Brief paper

Output-feedback sampled-data polynomial controller for nonlinear systems☆

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ABSTRACT

This paper presents the stability analysis and control synthesis for a sampled-data control system which consists of a nonlinear plant and an output-feedback sampled-data polynomial controller connected in a closed loop. The output-feedback sampled-data polynomial controller, which can be implemented by a microcontroller or a digital computer, is proposed to stabilize the nonlinear plant. Based on the Lyapunov stability theory, stability conditions in terms of sum of squares are obtained to guarantee the stability and to aid the design of a polynomial controller. A simulation example is given to demonstrate the effectiveness of the proposed control approach.

1. Introduction

Due to the rapid growth of computer technology, microcontrollers and digital computers can be available at low cost. A sampled-data controller implemented by a microcontroller or a digital computer can lower the implementation cost and time. However, due to the zero order hold (ZOH), the sampled-data controller holding the control signal constant during the sampling period introduces discontinuity to the system which complicates the system dynamics and makes the analysis difficult.

The stability of linear (Chen & Francis, 1991) and nonlinear (Monaco & Normand-Cyrot, 1995; Sontag, 1989) sampled-data control systems has been investigated for decades. Emulation is one of the methods for the design of sampled-data controllers. In general, a controller is designed based on the continuous-time plant, followed by a discretization process. Due to the difficulty in obtaining the exact discrete-time model of the nonlinear plant, an approximate discrete-time system model is employed to investigate the stability. Various stability properties were developed in Laila and Astolfi (2005), Laila and Nešić (2004), Laila, Nešić, and Teel (2002), Nešić and Angeli (2002), Nešić and Grüne (2005), Grüne, Worthmann, and Nešić (2008), Liu, Marquez, and Lin (2008), Mirkin (2007), Naghshtabrizi, Hespanha, and Teel (2006) and the references therein. The satisfaction of the stability properties guarantees the stability of the sampled-data control system formed by the continuous-time nonlinear plant and the sampled-data controller connected in a closed loop.

Recently, the stability of time-delay linear and nonlinear systems has been investigated based on the time-delay-dependent approach (Han, 2008; He, Wang, Xie, & Lin, 2007; He, Wu, She, & Liu, 2004; Xu & Lam, 2005) through the Lyapunov–Krasovskii functional and the time-delay-independent approach (Cao, Sun, & Cheng, 1998; He & Wu, 2003) through the Lyapunov–Razumikhin functional. Based on the time-delay control system analysis approach, the stability of sampled-data linear control systems was investigated by transforming the sampled-data control system as a continuous-time system with time-delayed control input (Fridman, Seuret, & Richard, 2004; Hu, Bai, Shi, & Wu, 2007). Followed by some matrix inequalities estimating the upper bounds of the cross terms, stability conditions in terms of linear matrix inequalities (LMIs) (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994) were obtained to guarantee the stability. A feasible solution of the LMI stability conditions can be found numerically using convex programming techniques.

In this paper, the stability of the sampled-data nonlinear systems is investigated based on the input delay approach (Fridman et al., 2004). An output feedback sampled-data (OFSD) polynomial controller is proposed for the control process. Compared to the full-state feedback controller, the output-feedback control (Lo & Lin, 2003) is more challenging as only the system output is available for feedback compensation. A sum-of-squares (SOS) approach is employed to carry out the stability analysis. Stability conditions in terms of SOS are derived based on the Lyapunov stability theory to guarantee the stability and facilitate the control synthesis. The SOS stability conditions can be solved numerically using the third-party Matlab toolbox SOSTOOLS (Prajna, Papachristodoulou, & Parrilo, 2002a), where the technical details of SOSTOOLS can be found

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in Prajna, Papachristodoulou, and Parrilo (2002b). The SOS techniques (Papachristodoulou & Prajna, 2005) generalizing the LMI-based approach were proposed by Prajna et al. (2002a). Instead of constant matrices in LMIs, all decision variables are polynomials in the SOS conditions.

Throughout this paper, the following notations are adopted (Prajna, Papachristodoulou, & Wu, 2004). The monomial vector $\hat{\mathbf{x}}(t)$ in $\mathbf{x}(t)$ of which each element is defined as $x_1^{d_1}(t)x_2^{d_2}(t)\cdots x_M^{d_M}(t)$ where d_i, $i = 1, 2, \ldots, M$, are nonnegative integers. The degree of a monomial is defined as $d = \sum_{i=1}^{M} d_i$. A polynomial $\mathbf{p}(\mathbf{x}(t))$ is defined as a finite linear combination of monomials with real coefficients. A polynomial $\mathbf{p}(\mathbf{x}(t))$ is an SOS if it can be written as $\sum_{j=1}^{r} \mathbf{q}_j(\mathbf{x}(t))^2$ where $\mathbf{q}_j(\mathbf{x}(t))$ is a polynomial and r is a nonzero positive integer. Hence, it can be seen that $\mathbf{p}(\mathbf{x}(t)) \geq 0$ if it is an SOS. It is stated in Papachristodoulou and Prajna (2005) that the polynomial $\mathbf{p}(\mathbf{x}(t))$ being an SOS can be represented in the form of $\hat{\mathbf{x}}(t)^T\mathbf{Q}\hat{\mathbf{x}}(t)$ where $\mathbf{Q}$ is a positive semi-definite matrix. The problem of finding a $\mathbf{Q}$ can be formulated as a semi-definite program (SDP). The SOSTOOLS can be used to find numerically the matrix $\mathbf{Q}$. To investigate the stability of the control systems, the Lyapunov function $V(t)$ is considered. The nonlinear system, say $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$, is asymptotically stable when $\dot{V}(t) = \frac{\partial V(t)}{\partial \mathbf{x}(t)}\mathbf{f}(\mathbf{x}(t)) < 0$ for $\mathbf{x}(t) \neq 0$. It is found that the construction of $V(t)$ formulated as SOS conditions can be done using semidefinite programming.

This paper is organized as follows. In Section 2, the nonlinear plant and an OFSD polynomial controller are introduced. In Section 3, the stability of the sampled-data control systems is investigated based on the Lyapunov stability theory. SOS stability conditions are obtained to guarantee the system stability. In Section 4, a simulation example is given to illustrate the merits of the proposed output feedback sampled-data control scheme. In Section 5, a conclusion is drawn.

2. Nonlinear plant and output-feedback sampled-data polynomial controller

A sampled-data control system consisting of a nonlinear plant and an OFSD polynomial controller connected in a closed loop is considered.

2.1. Nonlinear plant

A class of nonlinear systems in the following form is considered.

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}(\mathbf{x}(t))\mathbf{u}(t), \quad (1)$$

$$\mathbf{y}(t) = \mathbf{C}\hat{\mathbf{x}}(\mathbf{x}(t)), \quad (2)$$

where $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ is the system state vector, $\mathbf{A}(\mathbf{x}(t)) \in \Re^{n\times N}$ is the known system matrix, $\mathbf{B}(\mathbf{x}(t)) \in \Re^{n\times m}$ is the known input matrix, $\mathbf{u}(t) \in \Re^m$ is the control input vector, $\mathbf{y}(t) = [y_1(t), y_2(t), \ldots, y_l(t)]^T$ is the output vector, $\mathbf{C} \in \Re^{l\times N}$ is the constant system output matrix and $\hat{\mathbf{x}}(\mathbf{x}(t)) \in \Re^N$ is a vector with each entry as a unique monomial in $\mathbf{x}(t)$. It is assumed that $\hat{\mathbf{x}}(\mathbf{x}(t)) = \mathbf{0}$ iff $\mathbf{x}(t) = \mathbf{0}$.

2.2. Output-feedback sampled-data polynomial controller

An OFSD polynomial controller is defined as follows,

$$\begin{aligned}\mathbf{u}(t) &= \mathbf{G}\mathbf{y}(t_\gamma)\\ &= \mathbf{G}\mathbf{C}\hat{\mathbf{x}}(\mathbf{x}(t-\tau_s(t))), \quad t_\gamma \leq t < t_{\gamma+1},\ \gamma = 1, 2, \ldots, \infty \quad (3)\end{aligned}$$

where $\mathbf{G} \in \Re^{m\times l}$ is a constant feedback gain to be determined, $t_\gamma = \gamma h_s$ denotes the sampling instant, $h_s = t_{\gamma+1} - t_\gamma$ denotes the constant sampling period, $\tau_s(t) = t - t_\gamma < h_s$ for $t_\gamma \leq t < t_{\gamma+1}$. It should be noted that the control signal $\mathbf{u}(t) = \mathbf{u}(t_\gamma)$ is held constant for $t_\gamma \leq t < t_{\gamma+1}$.

Remark 1. The OFSD polynomial controller (3) becomes a full state-feedback one when $\mathbf{C}$ is a full rank matrix, for example, $\mathbf{C} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

3. Stability analysis

In this section, the sampled-data control system formed by the nonlinear plant (1) and the OFSD polynomial controller (3) is investigated. From (1) and (3), we have

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}(\mathbf{x}(t))\mathbf{G}\mathbf{C}\hat{\mathbf{x}}(\mathbf{x}(t-\tau_s(t))). \quad (4)$$

Definition 2 (*Khalil, 2002*). The equilibrium point $\mathbf{x}(t) = \mathbf{0}$ of (4) is asymptotically stable if it is stable and there exists δ such that $\|\mathbf{x}(0)\| < \delta \Rightarrow \lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$.

The stability of the sampled-data control system (4) is guaranteed by the following theorem.

Theorem 3. *The sampled-data control system (4), formed by the nonlinear plant in the form of (1) and (2) and the OFSD polynomial controller (3) connected in a closed loop, is guaranteed to be asymptotically stable if there exist predefined constant sampling period $h_s > 0$, predefined constant scalars ε_1, ε_2, ξ, $\varsigma_1 > 0$ and $\varsigma_2 > 0$, and the following decision variables, i.e., matrices $\mathbf{M} = \mathbf{M}^T \in \Re^{N\times N}$, $\mathbf{N} \in \Re^{m\times l}$, $\mathbf{X}_1 = \mathbf{X}_1^T = \begin{bmatrix}\mathbf{X}_{11} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_{22}\end{bmatrix} \in \Re^{N\times N}$, $\mathbf{X}_{11} = \mathbf{X}_{11}^T \in \Re^{l\times l}$ and $\mathbf{X}_{22} = \mathbf{X}_{22}^T \in \Re^{(N-l)\times(N-l)}$, polynomial matrices $\mathbf{U}(\mathbf{x}(t)) = \mathbf{U}(\mathbf{x}(t))^T \in \Re^{N\times N}$ and $\mathbf{W}(\mathbf{x}(t)) = \mathbf{W}(\mathbf{x}(t))^T \in \Re^{N\times N}$, and polynomial scalar $\varsigma_3(\mathbf{x}(t)) > 0$ such that the following SOS conditions are satisfied.*

$$\mathbf{r}(t)^T\Big(\mathbf{X}_1 - \varsigma_1\mathbf{I}\Big)\mathbf{r}(t) \text{ is SOS}, \quad (5)$$

$$\mathbf{r}(t)^T\Big(\mathbf{M} - \varsigma_2\mathbf{I}\Big)\mathbf{r}(t) \text{ is SOS}, \quad (6)$$

$$-\mathbf{s}(t)^T\Big(\hat{\boldsymbol{\Xi}}(\mathbf{x}(t)) + \varsigma_3(\mathbf{x}(t))\mathbf{I}\Big)\mathbf{s}(t) \text{ is SOS} \quad (7)$$

where $\mathbf{r}(t) \in \Re^N$ and $\mathbf{s}(t) \in \Re^{4N}$ are arbitrary vectors independent of $\mathbf{x}(t)$,

$$\hat{\boldsymbol{\Xi}}(\mathbf{x}(t)) = \begin{bmatrix}\boldsymbol{\Theta}(\mathbf{x}(t)) + \boldsymbol{\Theta}(\mathbf{x}(t))^T & * & *\\ h_s\boldsymbol{\Psi}(\mathbf{x}(t))^T & -h_s\mathbf{M} & *\\ h_s\boldsymbol{\Phi}(\mathbf{x}(t))^T & \mathbf{0} & -h_s(2\xi\mathbf{X}_1 - \xi^2\mathbf{M})\end{bmatrix},$$

$$\boldsymbol{\Theta}(\mathbf{x}(t)) = \begin{bmatrix}\boldsymbol{\Theta}_{11}(\mathbf{x}(t)) & \boldsymbol{\Theta}_{12}(\mathbf{x}(t))\\ \boldsymbol{\Theta}_{21}(\mathbf{x}(t)) & \boldsymbol{\Theta}_{22}(\mathbf{x}(t))\end{bmatrix},$$

$$\begin{aligned}\boldsymbol{\Theta}_{11}(\mathbf{x}(t)) &= \tilde{\mathbf{A}}(\mathbf{x}(t))\mathbf{X}_1 + \varepsilon_1\boldsymbol{\Upsilon}(\mathbf{x}(t)) + (1-\varepsilon_1)\mathbf{U}(\mathbf{x}(t))^T\\ &\quad + (1-\varepsilon_1)\varepsilon_1\mathbf{W}(\mathbf{x}(t))^T,\end{aligned}$$

$$\boldsymbol{\Theta}_{12}(\mathbf{x}(t)) = \varepsilon_2\boldsymbol{\Upsilon}(\mathbf{x}(t)) + (1-\varepsilon_1)\varepsilon_2\mathbf{W}(\mathbf{x}(t))^T,$$

$$\boldsymbol{\Theta}_{21}(\mathbf{x}(t)) = -\varepsilon_2\mathbf{U}(\mathbf{x}(t))^T - \varepsilon_1\varepsilon_2\mathbf{W}(\mathbf{x}(t))^T,$$

$$\boldsymbol{\Theta}_{22}(\mathbf{x}(t)) = -\varepsilon_2^2\mathbf{W}(\mathbf{x}(t))^T,$$

$$\boldsymbol{\Upsilon}(\mathbf{x}(t)) = \tilde{\mathbf{B}}(\mathbf{x}(t))\begin{bmatrix}\mathbf{N} & \mathbf{0}\end{bmatrix},$$

$$\boldsymbol{\Phi}(\mathbf{x}(t)) = \begin{bmatrix}\mathbf{U}(\mathbf{x}(t)) + \varepsilon_1\mathbf{W}(\mathbf{x}(t))\\ \varepsilon_2\mathbf{W}(\mathbf{x}(t))\end{bmatrix},$$

$$\boldsymbol{\Psi}(\mathbf{x}(t)) = \begin{bmatrix}\boldsymbol{\Psi}_1(\mathbf{x}(t))\\ \boldsymbol{\Psi}_2(\mathbf{x}(t))\end{bmatrix},$$

$$\boldsymbol{\Psi}_1(\mathbf{x}(t)) = \mathbf{X}_1\tilde{\mathbf{A}}(\mathbf{x}(t))^T + \varepsilon_1\boldsymbol{\Upsilon}(\mathbf{x}(t))^T,$$

$$\boldsymbol{\Psi}_2(\mathbf{x}(t)) = \varepsilon_2\boldsymbol{\Upsilon}(\mathbf{x}(t))^T,$$

$$\tilde{\mathbf{A}}(\mathbf{x}(t)) = \boldsymbol{\Gamma}^{-1}\mathbf{H}(\mathbf{x}(t))\mathbf{A}(\mathbf{x}(t))\boldsymbol{\Gamma},$$

$$\tilde{\mathbf{B}}(\mathbf{x}(t)) = \boldsymbol{\Gamma}^{-1}\mathbf{H}(\mathbf{x}(t))\mathbf{B}(\mathbf{x}(t)),$$

$\mathbf{H}(\mathbf{x}(t)) \in \Re^{N\times n}$ *with its* (i,j)*-th entry defined as* $H_{ij}(\mathbf{x}(t)) = \frac{\partial \hat{x}_i(\mathbf{x}(t))}{\partial x_j(t)}$, $i = 1,2,\ldots,N; j = 1,2,\ldots,n$, $\mathbf{\Gamma} = \left[\mathbf{C}^T(\mathbf{C}\mathbf{C}^T)^{-1} \quad ortc(\mathbf{C}^T)\right]$ *and* $ortc(\mathbf{C}^T)$ *is the orthogonal complement of* $\mathbf{C}^T$*, and the feedback gain of the sampled-data polynomial controller is defined as* $\mathbf{G} = \mathbf{N}\mathbf{X}_{11}^{-1}$.

Proof. From (4), we have,

$$\dot{\hat{\mathbf{x}}}(\mathbf{x}(t)) = \frac{\partial \hat{\mathbf{x}}(\mathbf{x}(t))}{\partial \mathbf{x}(t)}\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{H}(\mathbf{x}(t))\dot{\mathbf{x}}(t), \quad (8)$$

where $\mathbf{H}(\mathbf{x}(t))$ is defined in Theorem 3. From (4) and (8), denoting $\mathbf{z}(t) = \mathbf{\Gamma}^{-1}\hat{\mathbf{x}}(\mathbf{x}(t))$ where $\mathbf{\Gamma}$ is defined in Theorem 3, we have,

$$\begin{aligned}\dot{\mathbf{z}}(t) &= \mathbf{\Gamma}^{-1}\dot{\hat{\mathbf{x}}}(\mathbf{x}(t)) \\ &= \mathbf{\Gamma}^{-1}\mathbf{H}(\mathbf{x}(t))\Big(\mathbf{A}(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}(\mathbf{x}(t))\mathbf{G}\mathbf{C}\hat{\mathbf{x}}(\mathbf{x}(t-\tau_s(t)))\Big) \\ &= \tilde{\mathbf{A}}(\mathbf{x}(t))\mathbf{z}(t) + \tilde{\mathbf{B}}(\mathbf{x}(t))\mathbf{G}\mathbf{C}\mathbf{\Gamma}\mathbf{z}(t-\tau_s(t)), \quad (9)\end{aligned}$$

where $\tilde{\mathbf{A}}(\mathbf{x}(t))$ and $\tilde{\mathbf{B}}(\mathbf{x}(t))$ are defined in Theorem 3.

It can be seen that the stability of (9) implies that of (4). To investigate the stability of (9), we consider the following Lyapunov functional,

$$V(t) = \mathbf{z}(t)^T\mathbf{P}_1\mathbf{z}(t) + \int_{-h_s}^{0}\int_{t+\sigma}^{t}\dot{\mathbf{z}}(\varphi)^T\mathbf{R}\dot{\mathbf{z}}(\varphi)\mathrm{d}\varphi\mathrm{d}\sigma \quad (10)$$

where $\mathbf{P}_1 = \mathbf{P}_1^T \in \Re^{N\times N}$, $\mathbf{R} = \mathbf{R}^T \in \Re^{N\times N}$, $\mathbf{P}_1 > 0$ and $\mathbf{R} > 0$. From (9) and (10), denoting $\mathbf{h}(t) = \begin{bmatrix}\mathbf{z}(t) \\ \mathbf{z}(t-\tau_s(t))\end{bmatrix}$, we have,

$$\begin{aligned}\dot{V}(t) &= \mathbf{h}(t)^T(\mathbf{P}^T\mathbf{Q}(\mathbf{x}(t)) + \mathbf{Q}(\mathbf{x}(t))^T\mathbf{P})\mathbf{h}(t) \\ &\quad + h_s\dot{\mathbf{z}}(t)^T\mathbf{R}\dot{\mathbf{z}}(t) - \int_{t-h_s}^{t}\dot{\mathbf{z}}(\varphi)^T\mathbf{R}\dot{\mathbf{z}}(\varphi)\mathrm{d}\varphi \quad (11)\end{aligned}$$

where $\mathbf{P} = \begin{bmatrix}\mathbf{P}_1 & \mathbf{0} \\ \mathbf{P}_2 & \mathbf{P}_3\end{bmatrix}$, $\mathbf{P}_2 \in \Re^{N\times N}$ and $\mathbf{P}_3 \in \Re^{N\times N}$ are arbitrary matrices, and $\mathbf{Q}(\mathbf{x}(t)) = \begin{bmatrix}\tilde{\mathbf{A}}(\mathbf{x}(t)) & \tilde{\mathbf{B}}(\mathbf{x}(t))\mathbf{G}\mathbf{C}\mathbf{\Gamma} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}$.

To deal with the last term of (11), we consider the Newton–Leibniz rule and have $\int_{t-\tau_s(t)}^{t}\dot{\mathbf{z}}(\varphi)\mathrm{d}\varphi = \mathbf{z}(t) - \mathbf{z}(t-\tau_s(t))$. Then, the following inequality is considered to facilitate the stability analysis.

$$2\mathbf{h}(t)^T\begin{bmatrix}\mathbf{T}(\mathbf{x}(t)) \\ \mathbf{V}(\mathbf{x}(t))\end{bmatrix}\left(-\int_{t-\tau_s(t)}^{t}\dot{\mathbf{z}}(\varphi)\mathrm{d}\varphi + \mathbf{z}(t) - \mathbf{z}(t-\tau_s(t))\right) = \mathbf{0} \quad (12)$$

where $\mathbf{T}(\mathbf{x}(t)) \in \Re^{N\times N}$ and $\mathbf{V}(\mathbf{x}(t)) \in \Re^{N\times N}$ are arbitrary polynomial matrices. Based on the fact that $\tau_s(t) = t - t_\gamma < h_s$ and with (12), we consider the last term on the right hand side of (11) and have

$$\begin{aligned}-\int_{t-h_s}^{t}\dot{\mathbf{z}}(\varphi)^T\mathbf{R}\dot{\mathbf{z}}(\varphi)\mathrm{d}\varphi &\le -\int_{t-\tau_s(t)}^{t}\dot{\mathbf{z}}(\varphi)^T\mathbf{R}\dot{\mathbf{z}}(\varphi)\mathrm{d}\varphi \\ &\quad + 2\mathbf{h}(t)^T\begin{bmatrix}\mathbf{T}(\mathbf{x}(t)) \\ \mathbf{V}(\mathbf{x}(t))\end{bmatrix} \\ &\quad \times\left(-\int_{t-\tau_s(t)}^{t}\dot{\mathbf{z}}(\varphi)\mathrm{d}\varphi + \mathbf{z}(t) - \mathbf{z}(t-\tau_s(t))\right) \\ &\le 2\mathbf{h}(t)^T\begin{bmatrix}\mathbf{T}(\mathbf{x}(t)) \\ \mathbf{V}(\mathbf{x}(t))\end{bmatrix}\big(\mathbf{z}(t) - \mathbf{z}(t-\tau_s(t))\big) \\ &\quad + h_s\mathbf{h}(t)^T\begin{bmatrix}\mathbf{T}(\mathbf{x}(t)) \\ \mathbf{V}(\mathbf{x}(t))\end{bmatrix}\mathbf{R}^{-1}\begin{bmatrix}\mathbf{T}(\mathbf{x}(t)) \\ \mathbf{V}(\mathbf{x}(t))\end{bmatrix}^T\mathbf{h}(t). \quad (13)\end{aligned}$$

From (11) and (13), we have

$$\begin{aligned}\dot{V}(t) &\le \mathbf{h}(t)^T\Bigg(\mathbf{P}^T\mathbf{Q}(\mathbf{x}(t)) + \mathbf{Q}(\mathbf{x}(t))^T\mathbf{P} + h_s\begin{bmatrix}\mathbf{T}(\mathbf{x}(t)) \\ \mathbf{V}(\mathbf{x}(t))\end{bmatrix} \\ &\quad \times\mathbf{R}^{-1}\begin{bmatrix}\mathbf{T}(\mathbf{x}(t)) \\ \mathbf{V}(\mathbf{x}(t))\end{bmatrix}^T + \begin{bmatrix}\mathbf{T}(\mathbf{x}(t)) \\ \mathbf{V}(\mathbf{x}(t))\end{bmatrix}\begin{bmatrix}\mathbf{I} \\ -\mathbf{I}\end{bmatrix}^T \\ &\quad + \begin{bmatrix}\mathbf{I} \\ -\mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{T}(\mathbf{x}(t)) \\ \mathbf{V}(\mathbf{x}(t))\end{bmatrix}^T\Bigg)\mathbf{h}(t) + h_s\dot{\mathbf{z}}(t)^T\mathbf{R}\dot{\mathbf{z}}(t). \quad (14)\end{aligned}$$

Denote $\mathbf{X} = \mathbf{P}^{-1} = \begin{bmatrix}\mathbf{X}_1 & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{X}_3\end{bmatrix}$ where $\mathbf{X}_1 = \mathbf{X}_1^T \in \Re^{N\times N}$, $\mathbf{X}_1 > 0$, $\mathbf{X}_2 = \varepsilon_1\mathbf{X}_1 \in \Re^{N\times N}$, $\mathbf{X}_3 = \varepsilon_2\mathbf{X}_1 \in \Re^{N\times N}$, ε_1 and ε_2 are constant scalars to be determined. Denote $\mathbf{M} = \mathbf{R}^{-1} \in \Re^{N\times N}$, $\mathbf{U}(\mathbf{x}(t)) = \mathbf{X}_1\mathbf{T}(\mathbf{x}(t))\mathbf{X}_1 \in \Re^{N\times N}$, $\mathbf{W}(\mathbf{x}(t)) = \mathbf{X}_1\mathbf{V}(\mathbf{x}(t))\mathbf{X}_1 \in \Re^{N\times N}$, $\mathbf{Z}(t) = \begin{bmatrix}\mathbf{Z}_1(t) \\ \mathbf{Z}_2(t)\end{bmatrix} = \mathbf{X}^{-1}\begin{bmatrix}\mathbf{z}(t) \\ \mathbf{z}(t-\tau_s(t))\end{bmatrix}$ and $\dot{\mathbf{Z}}_1(t) = \mathbf{X}_1^{-1}\dot{\mathbf{z}}(t)$. From (9) and (14), we have

$$\dot{V}(t) \le \mathbf{Z}(t)^T\mathbf{\Xi}(\mathbf{x}(t))\mathbf{Z}(t) \quad (15)$$

where $\mathbf{\Xi}(\mathbf{x}(t)) = \mathbf{\Theta}(\mathbf{x}(t)) + \mathbf{\Theta}(\mathbf{x}(t))^T + h_s\mathbf{\Phi}(\mathbf{x}(t))\mathbf{X}_1^{-1}\mathbf{M}\mathbf{X}_1^{-1} \times \mathbf{\Phi}(\mathbf{x}(t))^T + h_s\mathbf{\Psi}(\mathbf{x}(t))\mathbf{M}^{-1}\mathbf{\Psi}(\mathbf{x}(t))^T$, and $\mathbf{\Theta}(\mathbf{x}(t))$, $\mathbf{\Phi}(\mathbf{x}(t))$ and $\mathbf{\Psi}(\mathbf{x}(t))$ are defined in Theorem 3.

To determine the feedback gain, as proposed in Lo and Lin (2003), we choose

$$\mathbf{X}_1 = \begin{bmatrix}\mathbf{X}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_{22}\end{bmatrix}, \quad (16)$$

where $\mathbf{X}_{11} = \mathbf{X}_{11}^T \in \Re^{l\times l}$ and $\mathbf{X}_{22} = \mathbf{X}_{22}^T \in \Re^{(N-l)\times(N-l)}$. Furthermore, we have

$$\mathbf{C}\mathbf{\Gamma} = \begin{bmatrix}\mathbf{I}_l & \mathbf{0}\end{bmatrix} \quad (17)$$

where $\mathbf{I}_l \in \Re^{l\times l}$ is the identity matrix. By expanding the terms in (15), we have the term $\mathbf{\Upsilon}(\mathbf{x}(t)) = \tilde{\mathbf{B}}(\mathbf{x}(t))\mathbf{G}\mathbf{C}\mathbf{\Gamma}\mathbf{X}_1$ which is nonlinear in $\mathbf{G}$ and $\mathbf{X}_1$ such that SOSTOOLS is not able to find a feasible solution numerically. To circumvent the problem, we choose the feedback gain as $\mathbf{G} = \mathbf{N}\mathbf{X}_{11}^{-1}$ where $\mathbf{N} \in \Re^{m\times l}$. From (17), we have,

$$\begin{aligned}\mathbf{\Upsilon}(\mathbf{x}(t)) &= \tilde{\mathbf{B}}(\mathbf{x}(t))\mathbf{N}\mathbf{X}_{11}^{-1}\mathbf{C}\mathbf{\Gamma}\mathbf{X}_1 \\ &= \tilde{\mathbf{B}}(\mathbf{x}(t))\mathbf{N}\mathbf{X}_{11}^{-1}\begin{bmatrix}\mathbf{I}_l & \mathbf{0}\end{bmatrix}\mathbf{X}_1 \\ &= \tilde{\mathbf{B}}(\mathbf{x}(t))\begin{bmatrix}\mathbf{N} & \mathbf{0}\end{bmatrix}, \quad (18)\end{aligned}$$

which is linear in $\mathbf{N}$ appearing in $\mathbf{\Xi}(\mathbf{x}(t))$. It can be seen from (15) that $\dot{V}(t) < 0$ when $\mathbf{\Xi}(\mathbf{x}(t)) < 0$ which implies the asymptotic stability of the sampled-data closed-loop system (4). Considering the inequality of $(\mathbf{X}_1 - \xi\mathbf{M})^T\mathbf{M}^{-1}(\mathbf{X}_1 - \xi\mathbf{M}) \ge 0$ where ξ is a constant scalar to be determined, we have

$$\mathbf{X}_1\mathbf{M}^{-1}\mathbf{X}_1 \ge 2\xi\mathbf{X}_1 - \xi^2\mathbf{M}. \quad (19)$$

By the Schur complement and with (19), $\mathbf{\Xi}(\mathbf{x}(t)) < 0$ is implied by the following inequality,

$$\begin{bmatrix}\mathbf{\Theta}(\mathbf{x}(t)) + \mathbf{\Theta}(\mathbf{x}(t))^T & * & * \\ h_s\mathbf{\Psi}(\mathbf{x}(t))^T & -h_s\mathbf{M} & * \\ h_s\mathbf{\Phi}(\mathbf{x}(t))^T & \mathbf{0} & -h_s(2\xi\mathbf{X}_1 - \xi^2\mathbf{M})\end{bmatrix} < 0 \quad (20)$$

where "*" denotes the transposed element at the corresponding entry. It can be seen that the SOS conditions (5)–(7) imply $\mathbf{X}_1 > 0$, $\mathbf{M} > 0$ and the inequality of (20), respectively. This completes the proof. □

Remark 4. It should be noted that the term $\mathbf{X}_1^{-1}\mathbf{M}\mathbf{X}_1^{-1}$ in (15) is nonlinear in $\mathbf{X}_1$. From inequality (19), we have the terms at the bottom right of (20), which are linear in $\mathbf{M}$ and $\mathbf{X}_1$, respectively. Consequently, SOSTOOLS can be applied to search for a feasible solution.

Remark 5. The above stability analysis is valid when $\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{X}_3 \end{bmatrix}$ is invertible. It can be seen that if there exists a solution to the SOS conditions (5) and (7), we have $\mathbf{X}_1 > 0$ and $\mathbf{X}_3 > 0$, which are sufficient conditions to guarantee that the matrix $\mathbf{X}$ is invertible.

Remark 6. It is not guaranteed that there exists a solution for Theorem 3. One necessary condition for Theorem 3 to have a solution is that the linearized model (1) at the origin is required to be controllable.

Remark 7. It should be noted that increasing the dimension of the system matrix and degree of monomials will increase the number of decision variables in SOSTOOLS. As a result, SOSTOOLS cannot solve the solution numerically when the number of decision variables is over the limit due to running out of memory. Given by an experiment, for a system with 7-by-7 system matrix, it is found that it will reach the limit of SOSTOOLS with about 155 decision variables.

Remark 8. For a given h_s satisfying the SOS stability conditions in Theorem 3, they also hold for any smaller sampling period.

4. Simulation example

A simulation example is given in this section to demonstrate the design procedure and merits of the proposed sampled-data control approach. Consider the nonlinear plant in the form of (1) and (2) with $\mathbf{A}(\mathbf{x}(t)) = \begin{bmatrix} a(\mathbf{x}(t)) & 0.2 & 0 \\ 1 & 0.3 & 2 - x_2(t) \end{bmatrix}$, $\mathbf{B}(\mathbf{x}(t)) = \begin{bmatrix} 1 \\ 2x_2(t) \end{bmatrix}$, $a(\mathbf{x}(t)) = -1 - 0.2(x_1(t) - 2)^2$, $\mathbf{C} = \begin{bmatrix} 0 & 5 & 10 \end{bmatrix}$, $\mathbf{x}(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T$, $\hat{\mathbf{x}}(\mathbf{x}(t)) = \begin{bmatrix} x_1(t) & x_2(t) & x_2(t)^2 \end{bmatrix}^T$ and $\boldsymbol{\Gamma} = \begin{bmatrix} 0.0000 & -0.4472 & -0.8944 \\ 0.0400 & 0.8000 & -0.4000 \\ 0.0800 & -0.4000 & 0.2000 \end{bmatrix}$. With SOSTOOLS (Prajna et al., 2002a), choosing $h_s = 0.002s$, $\varepsilon_1 = 500$, $\varepsilon_2 = 2000$, $\varsigma_1 = \varsigma_2 = \varsigma_3 = 0.001$ and $\xi = \sqrt{0.1}$, we found that the feedback gain $G = -0.6377$, $\mathbf{X}_1 = \begin{bmatrix} 0.1294 \times 10^{-2} & 0 \\ 0 & 0.6678 \times 10^{-3} \end{bmatrix}$ and $\mathbf{M} = \begin{bmatrix} 0.7366 & 0.5576 \times 10^{-4} \\ 0.5576 \times 10^{-4} & 0.7366 \end{bmatrix}$ which satisfy the SOS conditions in Theorem 3. The OFSD controller (3) is employed to control the nonlinear plant. The phase plot of $x_1(t)$ and $x_2(t)$ subject to various initial conditions is shown in Fig. 1. The control signal of the OFSD controller for the nonlinear system with the initial condition of $\mathbf{x}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ is shown in Fig. 2. It can be seen from Fig. 1 that the nonlinear plant can be stabilized successfully by the proposed OFSD controller. Furthermore, it can be seen from Fig. 2 that the control signal is a staircase function and with a constant level during the sampling period.

5. Conclusion

The stability of the nonlinear sampled-data control system consisting of a nonlinear plant and an output-feedback sampled-data (OFSD) polynomial controller has been investigated. The proposed OFSD polynomial controller uses the system output for feedback compensation. Due to the zero order hold, the control signal is kept constant during the sampling period. Consequently, the proposed OFSD polynomial controller can be implemented by a microcontroller or a digital computer to lower the implementation cost and time. Stability conditions in terms of sum of squares have been obtained based on the Lyapunov stability theory to aid the design of the OFSD polynomial controller. A simulation example has been given to illustrate the merits of the proposed control scheme.

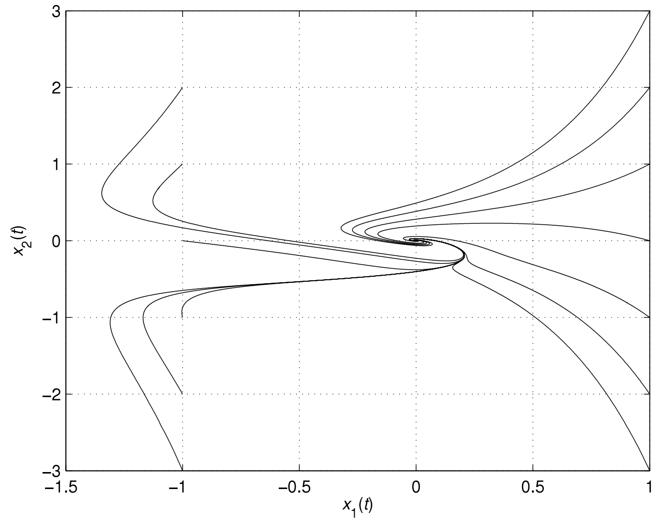

Fig. 1. Phase plot of $x_1(t)$ and $x_2(t)$.

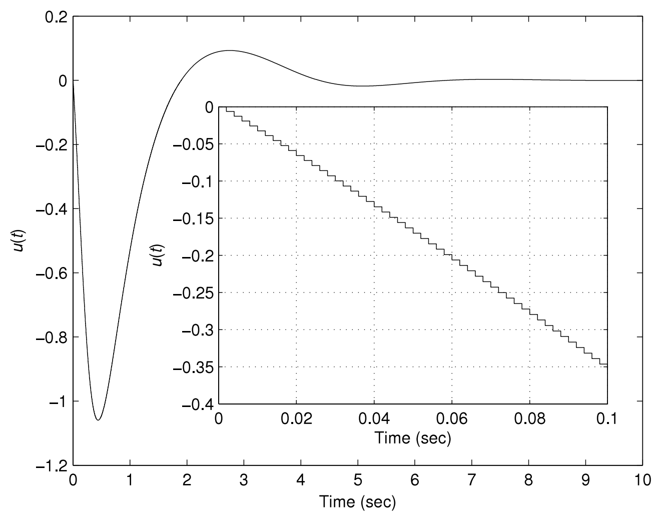

Fig. 2. Control signal $u(t)$.

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